Vectors

1- **Scalar magnitude and vector magnitude**

Scalar quantity: Any quantity defined by its magnitude and whose definition is complete is called scalar quantity, example: mass, energy, work….

Vector quantity: Any quantity not completely defined by its magnitude and which requires the knowledge of its direction is called vector quantity example: displacement, speed, force……

A vector is an oriented line segment. The vector is represented by a line segment AB of which A is the origin and B is the end, it is defined by:

-Its direction or support: is the line to which the line segment AB belongs (its support)

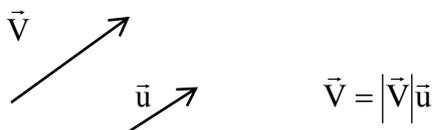- Its direction: is that of the displacement of a mobile from A to B

Its modulus: is the length of the line segment AB and it is denoted by AB or $\left|\overrightarrow{AB}\right|$.

The vector can also be denoted as follows $\vec{V}$

2-Unit vector

Definition: A unit vector $\vec{u}$ is a vector whose modulus is equal to unity $\left|\vec{u}\right| = 1$

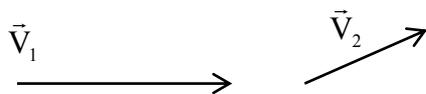Let be a vector $\vec{V}$ parallel to the unit vector $\vec{u}$

$$\vec{V} = \left|\vec{V}\right|\vec{u}$$

Let the vectors $\vec{V}_1$ and $\vec{V}_2$ be parallel to the unit vector $\vec{u}$.

$\vec{V}_1 = \left|\vec{V}_1\right|\vec{u}$ and $\vec{V}_2 = \left|\vec{V}_2\right|\vec{u}$
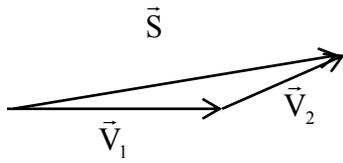
$\dfrac{\vec{V}_1}{\vec{V}_2} = \dfrac{\left|\vec{V}_1\right|}{\left|\vec{V}_2\right|} = k \Rightarrow \vec{V}_1 = k\vec{V}_2$ so the condition for 2 vectors to be parallel is:

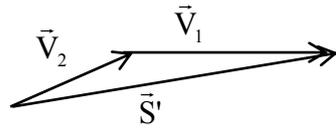$$\vec{V}_1 /\!/ \vec{V}_2 \Leftrightarrow \vec{V}_1 = k\vec{V}_2$$

3. Sum of vectors

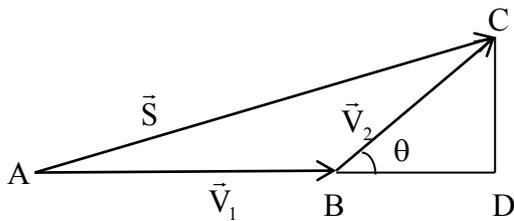

$$\vec{S} = \vec{V}_1 + \vec{V}_2$$

$$\vec{S}' = \vec{V}_2 + \vec{V}_1$$

$\vec{S} = \vec{S}'$ so $\vec{V}_1 + \vec{V}_2 = \vec{V}_2 + \vec{V}_1 \Rightarrow$ the vector sum is commutative.

$\vec{S}$ modulus

$$(AC)^2 = (AD)^2 + (DC)^2$$
$$AD = V_1 + V_2 \cos\theta$$
$$DC = V_2 \sin\theta$$
$$S^2 = (V_1 + V_2 \cos\theta)^2 + (V_2 \sin\theta)^2$$
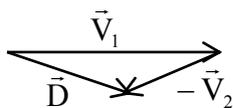$$= V_1^2 + V_2^2 \cos^2\theta + 2V_1V_2\cos\theta + V_2^2 \sin^2\theta$$
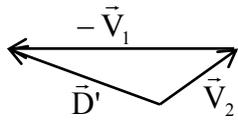$$S^2 = V_1^2 + V_2^2 + 2V_1V_2\cos\theta$$
$$S = \sqrt{V_1^2 + V_2^2 + 2V_1V_2\cos\theta}$$

Difference of two vectors

$$\vec{D} = \vec{V}_1 - \vec{V}_2 = \vec{V}_1 + \left(-\vec{V}_2\right)$$

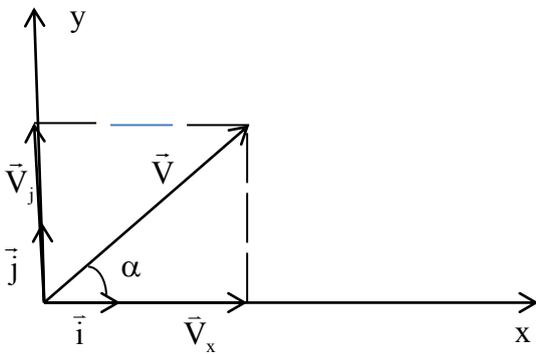
$$\vec{D}' = \vec{V}_2 - \vec{V}_1$$

$\vec{D} \neq \vec{D}'$, $\vec{V}_1 - \vec{V}_2 \neq \vec{V}_2 - \vec{V}_1$, so the difference of 2 vectors is not commutative.

$$D = \sqrt{V_1^2 + V_2^2 + 2V_1 V_2 \cos(\pi - \theta)}$$

4-Components of a vector

Any vector $\vec{V}$ can be written as a sum of two or more vectors. Any set of vectors whose sum gives the vector $\vec{V}$ is called components of this vector.

$$\vec{V} = \vec{V}_x + \vec{V}_y = V_x \vec{i} + V_y \vec{j}$$

$$\vec{V} \begin{vmatrix} V_x \\ V_y \end{vmatrix} \qquad \begin{aligned} V_x &= V \cos \alpha \\ V_y &= V \sin \alpha \end{aligned}$$

$$\vec{V} = V \cos \alpha \vec{i} + V \sin \alpha \vec{j} = V\left(\cos \alpha \vec{i} + \sin \alpha \vec{j}\right)$$
$$\vec{V} = |\vec{V}|\vec{u} \Rightarrow \vec{u} = \cos \alpha \vec{i} + \sin \alpha \vec{j}$$

Components of a vector in space

$$\vec{V} \begin{vmatrix} V_x \\ V_y \\ V_z \end{vmatrix} \qquad \begin{aligned} V_x &= V \sin \theta \cos \varphi \\ V_y &= V \sin \theta \sin \varphi \\ V_z &= V \sin \theta \end{aligned}$$
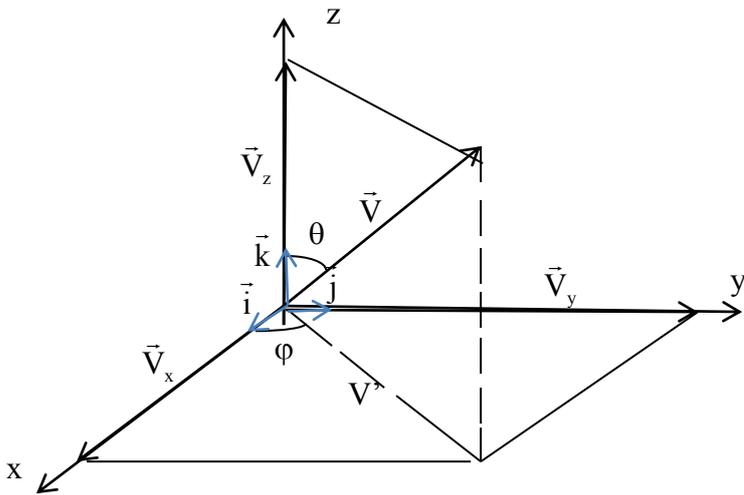
$$\vec{V} = V_x \vec{i} + V_y \vec{j} + V_z \vec{k}$$

$$V^2 = V'^2 + V_z^2$$

$$V^2 = V_x^2 + V_y^2 + V_z^2$$

$$V = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

Sum and difference of vectors

$$\vec{V}_1 \begin{vmatrix} V_{1x} \\ V_{1y} \\ V_{1z} \end{vmatrix} \qquad \vec{V}_2 \begin{vmatrix} V_{2x} \\ V_{2y} \\ V_{2z} \end{vmatrix}$$

$$\vec{V}_1 + \vec{V}_2 = (V_{1x} + V_{2x})\vec{i} + (V_{1y} + V_{2y})\vec{j} + (V_{1z} + V_{2z})\vec{k}$$

$$V_x = \sum_i V_{ix}$$

$$V_y = \sum_i V_{iy}$$

$$V_z = \sum_i V_{iz}$$

Same thing for a difference.

5- Direction cosines

Let α, β and γ be the angles that the vector $\vec{V}$ makes with the axes Ox, Oy and Oz respectively.

$$V_x = V \cos \alpha$$

$$V_y = V \cos \beta$$

$$V_z = V \cos \gamma$$

$$V^2 = V_x^2 + V_y^2 + V_z^2 = V^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \lambda) \Rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \lambda = 1$$

$\cos \alpha$, $\cos \beta$ and $\cos \gamma$ are the direction cosines of the vector $\vec{V}$

$$\vec{V} = |\vec{V}|\vec{u}$$
$$= V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$$
$$= V\cos\alpha\vec{i} + V\cos\beta\vec{j} + V\cos\gamma\vec{k}$$
$$= V(\cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}) \Rightarrow \vec{u} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$$

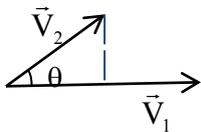The direction cosines are the components of the unit vector $\vec{u}$ of $\vec{V}$.

6-Scalar product

The scalar product of two vectors $\vec{V}_1$ and $\vec{V}_2$ denoted $\vec{V}_1.\vec{V}_2$, is a scalar quantity equal to the product of the modulus of the two vectors by the cosine of the angle formed by the two vectors.

$$\vec{V}_1.\vec{V}_2 = |\vec{V}_1||\vec{V}_2|\cos(\vec{V}_1 , \vec{V}_2) = |\vec{V}_1||\vec{V}_2|\cos\theta$$

$$\vec{V}_1.\vec{V}_2 > 0 \quad if \; |\theta| < \frac{\pi}{2}$$

$$\vec{V}_1.\vec{V}_2 < 0 \; if \; \frac{\pi}{2} < |\theta| < \pi$$

$$\vec{V}_1.\vec{V}_2 = V_1 \overline{proj}\, \vec{V}_2 \Big/ \vec{V}_1 = V_2 \overline{proj}\, \vec{V}_1 \Big/ \vec{V}_2 \; 2$$

Particular cases

$$\vec{V}.\vec{V} = V^2 \quad (\vec{V},\vec{V}) = 0$$

$$\vec{V}_1.\vec{V}_2 = 0 \Leftrightarrow (\vec{V}_1,\vec{V}_2) = \frac{\pi}{2}$$

Properties of the scalar product
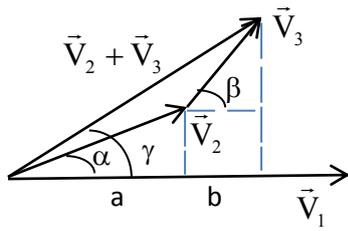
$$\vec{V}_1.\vec{V}_2 = V_1 V_2 \cos(\vec{V}_1 , \vec{V}_2)$$
$$\vec{V}_2.\vec{V}_1 = V_2 V_1 \cos(\vec{V}_2 , \vec{V}_1)$$
$$\cos(\vec{V}_1 , \vec{V}_2) = \cos(\vec{V}_2 , \vec{V}_1) \Rightarrow \vec{V}_1.\vec{V}_2 = \vec{V}_2.\vec{V}_1$$

The scalar product is commutative.

Distributivity

$$\vec{V}_1.(\vec{V}_2 + \vec{V}_3) = \vec{V}_1.\vec{V}_2 + \vec{V}_1.\vec{V}_3$$

$$\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2 \cos \alpha = V_1 a$$

$$\vec{V}_1 \cdot \vec{V}_3 = V_1 V_3 \cos \beta = V_1 b$$

$$\vec{V}_1 \cdot (\vec{V}_2 + \vec{V}_3) = \left|\vec{V}_1\right|\left|\vec{V}_2 + \vec{V}_3\right| \cos \gamma = V_1 (a + b)$$

$$\vec{V}_1 \cdot \vec{V}_2 + \vec{V}_1 \cdot \vec{V}_3 = V_1 (a + b)$$

$$\vec{V}_1 \cdot (\vec{V}_2 + \vec{V}_3) = \vec{V}_1 \cdot \vec{V}_2 + \vec{V}_1 \cdot \vec{V}_3$$

The scalar product is distributive

Unit vectors $\vec{i}$, $\vec{j}$, $\vec{k}$

$$\vec{i} \cdot \vec{i} = \vec{j} \cdot \vec{j} = \vec{k} \cdot \vec{k} = 1$$

$$\vec{i} \cdot \vec{j} = \vec{j} \cdot \vec{k} = \vec{k} \cdot \vec{i} = 0$$

Scalar product as a function of the components

$$\vec{V}_1 \begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} \qquad \vec{V}_2 \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix}$$

$$\vec{V}_1 \cdot \vec{V}_2 = \left(x_1 \vec{i} + y_1 \vec{j} + z_1 \vec{k}\right)\left(x_2 \vec{i} + y_2 \vec{j} + z_2 \vec{k}\right)$$

$$= x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\vec{V}_1 \cdot \vec{V}_2 = x_1 x_2 + y_1 y_2 + z_1 z_2$$

$$\vec{V} \cdot \vec{V} = x^2 + y^2 + z^2$$

$$V = \sqrt{x^2 + y^2 + z^2}$$

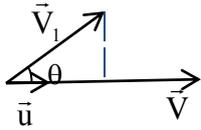## Direction cosines

$$\cos \alpha = \frac{\vec{V} \cdot \vec{i}}{\left|\vec{V}\right|}$$

$$\cos \beta = \frac{\vec{V} \cdot \vec{j}}{\left|\vec{V}\right|}$$

$$\cos \gamma = \frac{\vec{V} \cdot \vec{k}}{\left|\vec{V}\right|}$$
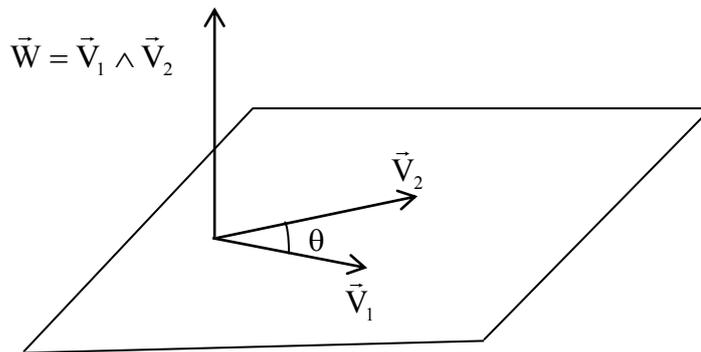
Projection of a vector

Projection of $\vec{V}_1$ onto $\vec{V}$

$$\overline{\mathrm{proj}}\vec{V}_1/\vec{V} = \vec{V}_1.\vec{u} \qquad \vec{u} = \frac{\vec{V}}{V}$$

7-Vector product

Definition: The vector product of two vectors $\vec{V}_1$ and $\vec{V}_2$ denoted $\vec{V}_1 \wedge \vec{V}_2$ is a vector perpendicular to the plane containing $\vec{V}_1$ and $\vec{V}_2$, and its direction is that of the displacement of a straight screw going $\vec{V}_1$ from to $\vec{V}_2$.

$$\left|\vec{V}_1 \wedge \vec{V}_2\right| = V_1 V_2 \sin\theta$$

$\vec{V}_1 \wedge \vec{V}_2 = -\vec{V}_2 \wedge \vec{V}_1$ The vector product is not commutative.

$\vec{V}_1 /\!/ \vec{V}_2 \Rightarrow \left(\vec{V}_1, \vec{V}_2\right) = 0$ so $\vec{V}_1 \wedge \vec{V}_2 = \vec{0}$

If the two vectors are parallel, their vector product is zero.

$$\vec{i} \wedge \vec{j} = -\vec{j} \wedge \vec{i} = \vec{k}$$
$$\vec{j} \wedge \vec{k} = -\vec{k} \wedge \vec{j} = \vec{i}$$
$$\vec{k} \wedge \vec{i} = -\vec{i} \wedge \vec{k} = \vec{j}$$
$$\vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = 0$$

The vector product is distributive with respect to the addition

$$\vec{V}_1 \wedge \left(\vec{V}_2 + \vec{V}_3\right) = \vec{V}_1 \wedge \vec{V}_2 + \vec{V}_1 \wedge \vec{V}_3$$
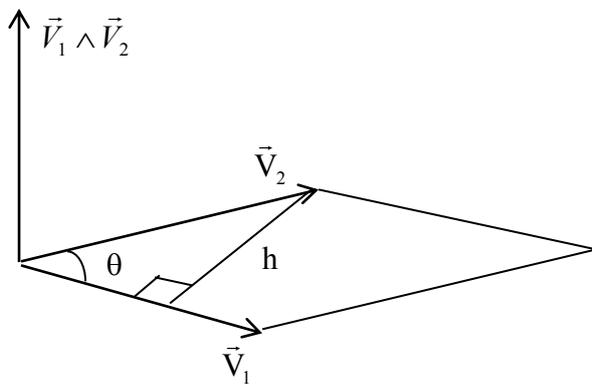
The vector product as a function of the components

$$\vec{V}_1 \begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} \qquad \vec{V}_2 \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix}$$

$$\vec{V}_1 \wedge \vec{V}_2 = \left(x_1\vec{i} + y_1\vec{j} + z_1\vec{k} \right) \wedge \left(x_2\vec{i} + y_2\vec{j} + z_2\vec{k} \right)$$
$$= \left(y_1 z_2 - y_2 z_1 \right)\vec{i} + \left(x_2 z_1 - x_1 z_2 \right)\vec{j} + \left(x_1 y_2 - x_2 y_1 \right)\vec{k}$$

Or using the determinant, it is written as follows:

$$\vec{V}_1 \wedge \vec{V}_2 \begin{vmatrix} + & - & + \\ \vec{i} & \vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\left| \vec{V}_1 \wedge \vec{V}_2 \right| = V_1 V_2 \sin\theta$$
$$= V_1 h$$

The modulus of the vector product represents the surface of the parallelogram formed by the two vectors.

8- Mixed product

We define the mixed product of three vectors $\vec{V}_1, \vec{V}_2$ and $\vec{V}_3$ by the scalar magnitude

$$\vec{\cdot} \left(\vec{V}_2 \wedge \vec{V}_3 \right)$$
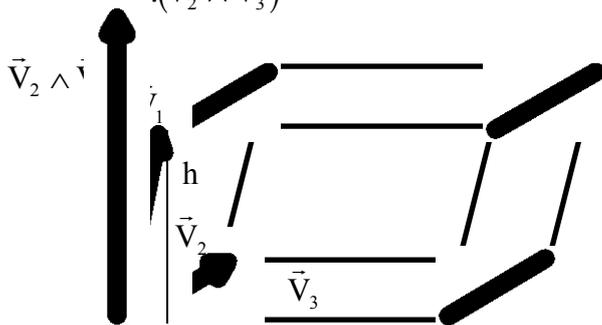
$\left| \vec{V}_2 \wedge \vec{V}_3 \right|$ is the surface of the parallelogram formed by $\vec{V}_2$ and $\vec{V}_3$

$$\vec{V}_1.(\vec{V}_2 \wedge \vec{V}_3) = \left| \vec{V}_2 \wedge \vec{V}_3 \right| \overline{\text{proj}} \vec{V}_1 / \vec{V}_2 \wedge \vec{V}_3$$
$$= S.h$$

So $\vec{V}_1.(\vec{V}_2 \wedge \vec{V}_3)$ represents the volume of the parallelepiped constructed by the vectors $\vec{V}_1, \vec{V}_2$

and $\vec{V}_3$

The mixed product is unchanged if we perform a circular permutation.

$$\vec{V}_1.(\vec{V}_2 \wedge \vec{V}_3) = \vec{V}_3.(\vec{V}_1 \wedge \vec{V}_2) = \vec{V}_2.(\vec{V}_3 \wedge \vec{V}_1)$$

Mixed product as a function of the components

$$\vec{V}_1 \begin{vmatrix} x_1 \\ y_1 \\ z_1 \end{vmatrix} \qquad \vec{V}_2 \begin{vmatrix} x_2 \\ y_2 \\ z_2 \end{vmatrix} \qquad \vec{V}_3 \begin{vmatrix} x_3 \\ y_3 \\ z_3 \end{vmatrix}$$

$$\vec{V}_1.(\vec{V}_2 \wedge \vec{V}_3) \begin{vmatrix} + & - & + \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= x_1(y_2 z_3 - y_3 z_2) + y_1(x_3 z_2 - x_2 z_3) + z_1(x_2 y_3 - x_3 y_2)$$

9- Moment of a vector

We define the moment of a vector $\overrightarrow{AB}$ with respect to any point O as follows:

$$\vec{M}_{\overrightarrow{AB}/O} = \overrightarrow{OA} \wedge \overrightarrow{AB}$$

The moment of a vector with respect to an axis Δ est:

$$M_{\overrightarrow{AB}/\Delta} = \vec{M}_{\overrightarrow{AB}/O}.\vec{u}$$

$\vec{u}$ is a unit vector carried by Δ

10-Vector derivatives

Let b a vector function, the derivative of the vector is defined as follows:

$$\frac{d\vec{V}}{dt} = \lim_{\Delta t \to 0} = \frac{\vec{V}(t + \Delta t) - \vec{V}(t)}{\Delta t}$$

$$\frac{d\vec{V}}{dt} = \frac{dV_x}{dt}\vec{i} + \frac{dV_y}{dt}\vec{j} + \frac{dV_z}{dt}\vec{k}$$

$$\vec{V} \begin{vmatrix} V_x \\ V_y \\ V_z \end{vmatrix} \qquad \frac{d\vec{V}}{dt} \begin{vmatrix} \dfrac{dV_x}{dt} \\[2mm] \dfrac{dV_y}{dt} \\[2mm] \dfrac{dV_z}{dt} \end{vmatrix}$$

The derivative of a vector is equal to the derivative of its components.

Properties

$$\frac{d}{dt}(\vec{V}_1 + \vec{V}_2) = \frac{d\vec{V}_1}{dt} + \frac{d\vec{V}_2}{dt}$$

$$\frac{d}{dt}(\lambda\vec{V}) = \frac{d\lambda}{dt}\vec{V} + \lambda\frac{d\vec{V}}{dt}$$

$$\frac{d}{dt}(\vec{V}_1 . \vec{V}_2) = \vec{V}_1 . \frac{d\vec{V}_2}{dt} + \vec{V}_2 . \frac{d\vec{V}_1}{dt}$$

$$\frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2) = \frac{d\vec{V}_1}{dt} \wedge \vec{V}_2 + \vec{V}_1 \wedge \frac{d\vec{V}_2}{dt}$$