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Depatment of Physics
LMD (2^{nd} **year Bachelor's degree in Physics. 2024-2025, Semester 3**)

Final exam in : **MATHS 3 (SERIES and O.D.E)**
Date : January 19, 2025 - Duration : 1h30.

EXERCISE 01 (08 points)

A) Let

$$\sum_{n\geq 1} u_n \quad , \quad u_n \geq 0 \tag{1}$$

is a converge numerical series.

1) Calculate : 0,5 0,5 1

$$\lim_{n\to+\infty} u_n \quad , \quad \lim_{n\to+\infty} v_n = \lim_{n\to+\infty} \frac{\sin u_n}{u_n} \quad , \quad \lim_{n\to+\infty} w_n = \lim_{n\to+\infty} u_n^2.v_n \tag{2}$$

2) Show that 1 1

$$\sum_{n\geq 1} v_n \text{ is divergent and } \sum_{n\geq 1} w_n \text{ is convergent.} \tag{3}$$

B) Let $x \geq 0$, we define the functions series by

$$\sum_{n\geq 1} u_n(x) = \sum_{n\geq 1} (-1)^n \frac{1}{n+x} \tag{4}$$

Prove that :

- the functions series is **point wise convergent** on $[0, +\infty[$. 1
- the functions series is **uniformly convergent** on $[0, +\infty[$. 1
- the functions series is not **normally convergent** on $[0, +\infty[$. 1 + 1 not Abs

EXERCISE 02 (7 points)

1) Determine if the following improper integrals converges or diverges :

$$1 \qquad I = \int_1^{+\infty} \frac{(\cos t)^2}{1+t^2} dt \quad , \quad J = \int_0^{+\infty} \frac{\sin t}{t} dt \qquad 2$$

2) Calculate the values of the following integrals :

$$\int_1^{+\infty} \frac{1}{1+t^2} dt \quad and \quad J$$

1

3) Deduce : 1 1

$$|I| \leq \frac{\pi}{4} \quad \text{and the value of} \quad \int_0^{+\infty} \frac{\sin t^{2025}}{t} dt. \qquad 1$$

EXERCISE 03 (5 points)

1)- Find the coefficients a, b and c such that

$$\frac{2}{(p^2+4)(p+1)} = \frac{ap+b}{p^2+4} + \frac{c}{p+1} \qquad 0,5 + 0,5 + 0,5 = 1,5$$

2)- By the Laplace transform solve the following differential equation :

$$y' + y = \sin 2t \quad , \quad avec \quad y(0) = 2. \qquad 3,5$$

Series and O.D.E.   S3 / 2024-2025

Solution EX.1 : 8 points

(A) 1°). $\sum_{n=1}^{\infty} u_n$, $u_n \geq 0$ is a CV series $\Rightarrow \lim_{n\to+\infty} u_n = 0$

.. $\lim_{n\to+\infty} v_n = \lim_{n\to+\infty} \frac{\sin u_n}{u_n} = \lim_{u_n\to 0} \frac{\sin u_n}{u_n} = 1$

... $\lim_{n\to+\infty} w_n = \lim_{n\to+\infty} u_n^2 v_n = \lim_{n\to+\infty} u_n \sin u_n$

$= \lim_{u_n\to 0} u_n \sin u_n = 0$ because $0 \leq |u_n \sin u_n| \leq u_n \to 0$

or $-u_n \leq w_n \leq u_n \to 0$, $0 \leftarrow -u_n$

$\Rightarrow \lim_{n\to+\infty} w_n = 0$

2°). $\lim_{n\to+\infty} v_n = 1 \neq 0 \Rightarrow \sum v_n$ div.

.. we have : $|w_n| = |u_n \sin u_n| \leq u_n$

and $\sum u_n$ CV $\Rightarrow \sum w_n$ is Abs.CV $\Rightarrow$ CV.

(B) * <u>point-wise CV</u> :

Let $x \in [0, +\infty[$: then $\sum (-1)^n \frac{1}{n+x}$ is an alterning numerical series

1) $\lim_{n\to+\infty} |u_n(x)| = \lim_{n\to+\infty} \frac{1}{n+x} = 0$ ✓

2) $|u_n(x)| = \frac{1}{n+x}$ is decreasing on $[0, +\infty[$

because $\left(\frac{1}{n+x}\right)' = \frac{-1}{(n+x)^2} < 0$ ✓

~~so~~ by Leibniz theorem the series

$\sum_{n\geq 1} (-1)^n \frac{1}{n+x}$ is CV

* <u>Uniform-CV</u>: the series $\sum (-1)^n \frac{1}{n+x}$ is an Alterning CV series then

for all $x \in [0, +\infty[$: $|R_n(x)| \leq |u_{n+1}(x)| = \frac{1}{n+1+x} \leq \frac{1}{n+1} \to 0$

$\Rightarrow \lim_{n\to+\infty} \sup_{x\in[0,+\infty[} |R_n(x)| = 0 \Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{1}{n+x}$ is Unif. CV on $[0, +\infty[$

∘∘∘ The function series is not normally convergent on $[0,+\infty[$

because is not Absolutely cv on $[0,+\infty[$.

So for $x \in [0,+\infty[$

$$\sum_{n\geq 1} |u_n(x)| = \sum \frac{1}{n+x} \quad \text{we have: } \frac{1}{n+x} \underset{+\infty}{\sim} \frac{1}{n}$$

$$\sum_{n=1}^{+\infty} \frac{1}{n} \text{ div} \Rightarrow \sum_{n=1}^{+\infty} |u_n(x)| = \sum_{n=1}^{\infty} \frac{1}{n+x} \text{ div}$$

$$\Rightarrow \sum u_n(x) = \sum (-1)^n \frac{1}{n+x} \text{ is not abs-cv on } [0,+\infty[.$$

Sol. EX02:

1°) $I = \int_1^{\to+\infty} \frac{(\cos t)^2}{1+t^2}\,dt$ is improper because we integrate on unbunded interval $[1,+\infty[$

$$\frac{(\cos t)^2}{1+t^2} \leq \frac{1}{1+t^2} \leq \frac{1}{t^2} \qquad \left(\begin{array}{l} |\cos t| \leq 1 \\ (\cos t)^2 \leq |\cos t| \leq 1 \end{array}\right)$$

and $\int_1^{+\infty} \frac{1}{t^2}\,dt$ cv $\left(\begin{array}{c}\text{p-series} \\ p=2>1\end{array}\right) \Rightarrow \int_1^{+\infty} \frac{(\cos t)^2}{1+t^2}\,dt$ cv

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•• $J = \int_{\to 0}^{\to +\infty} \frac{\sin t}{t}\,dt$ improper at two borne

Then we study two improper integrals: J_1 and J_2

▪ $J_1 = \int_{\to 0}^{1} \frac{\sin t}{t}\,dt$ we have: $\lim_{t\to 0^+} \frac{\sin t}{t} = 1$ finit

$$\Rightarrow J_1 = \int_0^1 \frac{\sin t}{t}\,dt \text{ is CV.}$$

▫ $J_2 = \int_1^{\to+\infty} \frac{\sin t}{t}\,dt$ we integrate by parts:

$$\begin{cases} u = \frac{1}{t} \\ v' = \sin t \end{cases} \Rightarrow \int_1^{+\infty} \frac{\sin t}{t}\,dt = \left[-\frac{\cos t}{t}\right]_1^{+\infty} - \int_1^{+\infty} \frac{\cos t}{t^2}\,dt$$

$$= \left[\underbrace{\lim_{t\to+\infty} -\frac{\cos t}{t}}_{=0} - \left(-\frac{\cos 1}{1}\right)\right] - \int_1^{+\infty} \frac{\cos t}{t^2}\,dt$$

$$= \underbrace{\cos 1}_{\text{finit}} - \int_1^{+\infty} \frac{\cos t}{t^2}\,dt$$

and $\int_1^{+\infty} \frac{\cos t}{t^2}\,dt$ abs. cv $\Rightarrow$ c

$\left|\frac{\cos t}{t^2}\right| \leq \frac{1}{t^2}$

and $\int_1^{+\infty} \frac{1}{t^2}\, dt$ cv (Riemann integrals $p = 2 > 1$)

$\Rightarrow \int_1^{+\infty} \frac{\cos t}{t^2}\, dt$ Abs. cv $\Rightarrow$ cv

$\Rightarrow J_2 = \int_1^{+\infty} \frac{\sin t}{t}\, dt$ cv

~~have~~ J_1 and J_2 cv $\Rightarrow$ $\boxed{J \text{ cv}}$

2) • $\int_1^{+\infty} \frac{1}{1+t^2}\, dt = \left[\operatorname{arctg} t\right]_1^{+\infty} = \operatorname{arctg}(+\infty) - \operatorname{arctg}(1)$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$

•• we calculate $J = \int_0^{+\infty} \frac{\sin t}{t}\, dt$

let $J(b) = \int_0^{+\infty} e^{-bt} \cdot \frac{\sin t}{t}\, dt$ for $\boxed{b = 0}$ $J(0) = J = \int_0^{+\infty} \frac{\sin t}{t}\, dt$

$$\frac{dJ(b)}{db} = \frac{d}{db} \int_0^{+\infty} e^{-bt} \frac{\sin t}{t}\, dt = \int_0^{+\infty} -e^{-bt} \sin t\, dt$$

$J'(b) = -\int_0^{+\infty} e^{-bt} \sin t\, dt$ by parts:

$$= -\left[\left[e^{-bt} \cdot (-\cos t)\right]_0^{+\infty} - \int_0^{+\infty} -b e^{-bt} (-\cos t)\, dt\right]$$

$$= -\left[0 + 1\right] + b \int_0^{+\infty} e^{-bt} \cos t\, dt$$

$$= -1 + b \int_0^{+\infty} e^{-bt} \cos t\, dt \quad \text{by parts}$$

$$= -1 + b\left[\left[e^{-bt} \sin t\right]_0^{+\infty} - \int_0^{+\infty} -b e^{-bt} \sin t\, dt\right]$$

$$J'(b) = -1 + b\left[[0-0] + b\int_0^{+\infty} e^{-bt}\sin t\,dt\right]$$

$$= -1 + b^2\int_0^{+\infty} e^{-bt}\sin t\,dt = -1 + b^2\left(-J'(b)\right)$$

$$J'(b) = -1 - b^2 J'(b)$$

$$J'(b) + b^2 J'(b) = -1 \Rightarrow J'(b)\left(1+b^2\right) = -1$$

$$\Rightarrow J'(b) = -\frac{1}{1+b^2}$$ by integration

$$J(b) = \int -\frac{1}{1+b^2}\,db = -\text{Arctg}(b) + C$$

for $b \to +\infty$: $J(+\infty) = -\text{Arctg}(+\infty) + C$

$\left(J(b) = \int_0^{+\infty} e^{-bt}\frac{\sin t}{t}\,dt\right)$ $\quad 0 = -\frac{\pi}{2} + C \Rightarrow \boxed{C = \frac{\pi}{2}}$

$$\Rightarrow J(b) = -\text{Arctg}(b) + \frac{\pi}{2}$$

for $\boxed{b=0}$: $J(0) = J = -\text{Arctg}(0) + \frac{\pi}{2}$

$$= -0 + \frac{\pi}{2}$$

$$\Rightarrow \boxed{J = \int_0^{+\infty}\frac{\sin t}{t}\,dt = \frac{\pi}{2}}$$

3) • $|I| = \left|\int_1^{+\infty}\frac{(\cos t)^2}{1+t^2}\,dt\right| \le \int_1^{+\infty}\left|\frac{(\cos t)^2}{1+t^2}\right|dt \le \int_1^{+\infty}\frac{1}{1+t^2}\,dt = \frac{\pi}{4}$

$$\Rightarrow \boxed{|I| \le \frac{\pi}{4}}$$

•• $\int_0^{+\infty}\frac{\sin t^{2025}}{t}\,dt$ $\quad$ on pose : $T = t^{2025} \Rightarrow dT = 2025\,t^{2024}\,dt$

$t = 0 \Rightarrow T = 0$

$t \to +\infty \Rightarrow T \to +\infty$

$$= \int_0^{+\infty}\frac{\sin T}{t}\cdot\frac{dT}{2025\,t^{2024}} = \int_0^{+\infty}\frac{\sin T}{2025\cdot t^{2025}}\,dT = \frac{1}{2025}\cdot\int_0^{+\infty}\frac{\sin}{T}$$

$= \frac{1}{2}$

Sol. EX03

Finding the Coefficients a, b, c:

$$\frac{2}{(p^2+4)(p+1)} = \frac{ap+b}{p^2+4} + \frac{c}{p+1}$$

$$\Leftrightarrow \quad 2 = (ap+b)(p+1) + c(p^2+4)$$

for: $p=-1$: $\quad 2 = 0 + c(1+4) \Rightarrow \boxed{c = \frac{2}{5}}$

for $p=0$: $\quad 2 = b + 4c \Rightarrow b = 2 - 4.c = 2 - 4.\frac{2}{5}$

$$\Rightarrow b = \frac{10-8}{5} = \frac{2}{5} \Rightarrow \boxed{b = \frac{2}{5}}$$

for $p=1$

$$2 = (a+b)2 + c.5$$

$$2 = (a+b).2 + \frac{2}{5}.5$$

$$2 = (a+b).2 + 2 \Rightarrow 2(a+b) = 0 \Rightarrow \boxed{a = -b}$$

$$\Rightarrow \boxed{a = -\frac{2}{5}}$$

2) Solving the differential equation by Laplace-Transformation

$$y' + y = \sin 2t \qquad y(0) = 2$$

$$\Rightarrow \mathcal{L}(y' + y) = \mathcal{L}(\sin 2t)$$

$$p\mathcal{L}(y) - y(0) + \mathcal{L}(y) = \frac{2}{p^2+2^2}$$

$$(p+1)\mathcal{L}(y) - 2 = \frac{2}{p^2+4} \Leftrightarrow (p+1)\mathcal{L}(y) = \frac{2}{p^2+4} + 2$$

$$\mathcal{L}(y) = \frac{2}{(p^2+4)(p+1)} + \frac{2}{p+1} \Rightarrow y = \mathcal{L}^{-1}\left(\frac{2}{(p^2+4)(p+1)} + \frac{2}{p+1}\right)$$

for Question (1) :

$$\mathcal{L} \quad \frac{2}{(p^2+4)(p+1)} = \frac{-\frac{2}{5}p + \frac{2}{5}}{p^2+4} + \frac{\frac{2}{5}}{p+1}$$

$$= \frac{2}{5}\left(\frac{-p+1}{p^2+4}\right) + \frac{2}{5}\cdot\frac{1}{p+1}$$

$$= \frac{2}{5}\left[-\frac{p}{p^2+2^2} + \frac{1}{p^2+2^2}\right] + \frac{2}{5}\,\frac{1}{p+1}$$

then

$$\mathcal{L}(y) = \frac{2}{5}\left[-\frac{p}{p^2+2^2} + \frac{1}{p^2+2^2}\right] + \frac{2}{5}\cdot\frac{1}{p+1} + \frac{2}{p+1}$$

$$= \frac{2}{5}\left[-\frac{p}{p^2+2^2}\right] + \frac{1}{5}\,\frac{2}{p^2+2^2} + \frac{12}{5}\cdot\frac{1}{p+1}$$

$$\Rightarrow \mathcal{L}(y) = -\frac{2}{5}\left[\frac{p}{p^2+2^2}\right] + \frac{1}{5}\left[\frac{2}{p^2+2^2}\right] + \frac{12}{5}\cdot\left(\frac{1}{p+1}\right)$$

by inverse Laplace Transformer we get the solution:

$$y = -\frac{2}{5}\mathcal{L}^{-1}\left(\frac{p}{p^2+2^2}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{2}{p^2+2^2}\right) + \frac{12}{5}\mathcal{L}^{-1}\left(\frac{1}{p+1}\right)$$

$$\boxed{y = -\frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t + \frac{12}{5}e^{-t}}$$