ليسانس فيزياء أساسية
الطرق الرياضية للفيزياء
الإمتحان النهائي

التمرين الأول (3 × 2 = 6 ن)

احسب التكاملات التالية :

$$\textbf{i)}\ \int_0^{\infty} \frac{\mathrm{e}^{-3x}}{\sqrt{x}}\,\mathrm{d}x \qquad \textbf{ii)}\ \int_0^{1} \frac{(1-x)^2}{\sqrt{x}}\,\mathrm{d}x \qquad \textbf{iii)}\ \int_0^{1} x^3(\ln x)^2\,\mathrm{d}x$$

التمرين الثاني (1 + 4 + 1.5 + 1.5 = 8 ن)

لتكن الدالة الدورية التالية (دورها يساوي 2π) :

$$f(x) = x^2 \qquad (-\pi \le x \le \pi)$$

1) ارسم $f(x)$ في المجال $(-3\pi,\ 3\pi)$.

2) عين معاملات "فورييه" (Fourier) المتعلقة بهذه الدالة .

3) اكتب متسلسلة "فورييه" .

4) استنتج الجمع التالي :

$$S = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

التمرين الثالث (2 + 4 = 6 ن)

لتكن الدوال التالية :

$$\varphi_1(x) = x \qquad \varphi_2(x) = x^2 \qquad \varphi_3(x) = x + ax^2 + bx^3$$

1) برهن أن الدالتين $f_1(x)$ و $f_2(x)$ متعامدتان في المجال $(-1,\ 1)$.

2) احسب الثابتين a و b بحيث تكون $f_3(x)$ متعامدة مع $f_1(x)$ و $f_2(x)$ في نفس المجال .

معطيات

(1)

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}\mathrm{d}t \qquad B(x,y) = \int_0^{1} t^{x-1}(1-t)^{y-1}\mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \qquad \Gamma(1/2) = \sqrt{\pi}$$

(2)

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\mathrm{d}x \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\mathrm{d}x \qquad \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)$$

Ex. 1 : 6 pts

i) $\int_0^\infty \frac{e^{-3x}}{\sqrt{x}}\,dx = \int_0^\infty x^{-1/2} e^{-3x}\,dx$

$t = 3x \Rightarrow x = \frac{t}{3} \qquad dx = \frac{dt}{3}$

$= \frac{\sqrt{3}}{3}\int_0^\infty t^{-1/2} e^{-t}\,dt = \frac{\sqrt{3}\,\Gamma(1/2)}{3}$

$= \frac{\sqrt{3\pi}}{3} = \sqrt{\frac{\pi}{3}}$ /2

ii) $\int_0^1 \frac{(1-x)^2}{\sqrt{x}}\,dx = \int_0^1 x^{-1/2}(1-x)^2\,dx$

$= B\left(\frac{1}{2},3\right) = \frac{\Gamma(1/2)\,\Gamma(3)}{\Gamma(7/2)}$

$= \frac{\Gamma(1/2)\times 2!}{5/2\times 3/2\times 1/2\times \Gamma(1/2)} = \frac{16}{15}$ /2

iii) $\ln x = -u$

$\Rightarrow x = e^{-u} \qquad dx = -e^{-u}\,du$

$\int_0^1 x^3(\ln x)^2\,dx = -\int_\infty^0 e^{-3u}u^2 e^{-u}\,du$

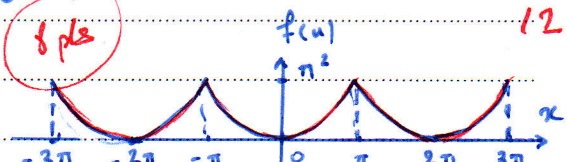
$= \int_0^\infty u^2 e^{-4u}\,du$

$4u = t \Rightarrow u = \frac{t}{4} \qquad du = \frac{dt}{4}$

$= \frac{1}{64}\int_0^\infty t^2 e^{-t}\,dt = \frac{\Gamma(3)}{64} = \frac{2!}{64} = \frac{1}{32}$ /2

Ex. 2 : 8 pts

1)



$f(x)$: paire $\qquad 2L = 2\pi \Rightarrow L = \pi$ /1

2) $b_n = 0 \quad \forall n$ /0.5

$a_n = \frac{2}{\pi}\int_0^\pi x^2\cos nx\,dx$

$u = x^2 \Rightarrow u' = 2x$

$v' = \cos nx \Rightarrow v = \frac{\sin nx}{n}$

$a_n = \frac{2}{\pi}\left\{ \overset{0}{x^2\frac{\sin nx}{n}\Big|_0^\pi} - \frac{2}{n}\int_0^\pi x\sin nx\,dx \right\}$

$= -\frac{4}{\pi n}\int_0^\pi x\sin nx\,dx \qquad (n\neq 0)$ /1.5

$u = x \Rightarrow u' = 1$

$v' = \sin nx \Rightarrow v = -\frac{\cos nx}{n}$

$a_n = -\frac{4}{\pi n}\left\{ -\frac{x\cos nx}{n}\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx \right\}$

$= \frac{4}{n^2}\cos n\pi - \frac{4}{\pi n^3}\overset{0}{\cancel{\sin nx}}\Big|_0^\pi$

$= \frac{4\cos n\pi}{n^2} \qquad (n\neq 0)$ /1.5

$a_0 = \frac{2}{\pi}\int_0^\pi x^2\,dx = \frac{2}{\pi}\,\frac{x^3}{3}\Big|_0^\pi = \frac{2\pi^2}{3}$ /0.5

3) $\frac{a_0}{2} + \sum_{n=1}^\infty \left(a_n\cos nx + b_n\sin nx\right)$

$= \frac{\pi^2}{3} + 4\sum_{n=1}^\infty \left(\frac{\cos n\pi}{n^2}\cos nx\right)$

$= \frac{\pi^2}{3} + 4\left(-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \dots\right)$

$= \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \dots\right)$ /1.5

4) $x = 0 \Rightarrow f(0) = 0$

$0 = \frac{\pi^2}{3} - 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots\right)$

$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$ /1.5

Ex. 3 6 pts

1) $\int_{-1}^1 \varphi_1(x)\varphi_2(x)\,dx = \int_{-1}^1 x^3\,dx$

$= \left[\frac{x^4}{4}\right]_{-1}^1 = \frac{1}{4} - \frac{1}{4} = 0 \Rightarrow \varphi_1 \perp \varphi_2$ /2

2) $\int_{-1}^1 \varphi_1(x)\varphi_3(x)\,dx = 0$

$= \int_{-1}^1 x(x + ax^2 + bx^3)\,dx$

$= \int_{-1}^1 (x^2 + ax^3 + bx^4)\,dx$

$$= \frac{x^3}{3}\Big|_{-1}^{1} + a\frac{x^4}{4}\Big|_{-1}^{1} + b\frac{x^5}{5}\Big|_{-1}^{1}$$

$$= \frac{2}{3} + 0 + \frac{2b}{5} = 0$$

$$\Rightarrow \quad \frac{2b}{5} = -\frac{2}{3} \Rightarrow b = -\frac{5}{3}$$

/2

$$\int_{-1}^{1} \varphi_2(x)\,\varphi_3(x) = 0$$

$$= \int_{-1}^{1} x^2(x + ax^2 + bx^3)\,dx$$

$$= \int_{-1}^{1} (x^3 + ax^4 + bx^5)\,dx$$

$$= \frac{x^4}{4}\Big|_{-1}^{1} + \frac{ax^5}{5}\Big|_{-1}^{1} + \frac{bx^6}{6}\Big|_{-1}^{1}$$

$$= 0 + \frac{2a}{5} + 0 = 0 \Rightarrow a = 0$$

$$\varphi_3(x) = x - \frac{5}{3}x^3$$

/2