Exercice 1 [2+5+2+2=11 pts]

Soit la fonction périodique (de période égale à 2π) suivante :

$$f(x) = \begin{cases} 0 & -\pi < x < 0 \\ \pi - x & 0 < x < \pi \end{cases}$$

a) Tracer $f(x)$ entre -2π et 2π.

b) Calculer les coefficients de Fourier a_n et b_n correspondants à cette fonction.

c) Écrire la série de Fourier correspondante.

d) En déduire la somme suivante :

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Exercice 2 [1+4+4=9 pts]

Soit la fonction suivante :

$$f(x) = \begin{cases} 1 - x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

a) Tracer cette fonction pour $x \ge 0$.

b) Déterminer la transformée de Fourier en sinus de cette fonction.

c) En utilisant l'identité de Parseval, calculer l'intégrale suivante :

$$\int_0^{+\infty} \left(\frac{\alpha - \sin\alpha}{\alpha^2} \right)^2 \mathrm{d}\alpha$$

(1)

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} \mathrm{d}x \qquad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\frac{n\pi x}{L} \mathrm{d}x$$

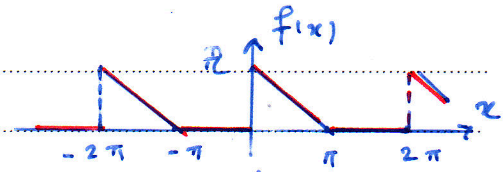
$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

(2)

$$F_S(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin\alpha x \, \mathrm{d}x \qquad \int_0^{\infty} [f(x)]^2 \, \mathrm{d}x = \frac{2}{\pi} \int_0^{\infty} [F_S(\alpha)]^2 \, \mathrm{d}\alpha$$

Ex. 1

a)



$2L = 2\pi \Rightarrow L = \pi$

b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$ /2

$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\cos nx\, dx$

$u = \pi - x \Rightarrow u' = -1$

$v' = \cos nx \Rightarrow v = \frac{\sin nx}{n}$ $(n \neq 0)$

$a_n = \frac{1}{\pi}\left\{ \cancel{\frac{(\pi - x)\sin nx}{n}\Big|_0^{\pi}} + \frac{1}{n}\int_0^{\pi}\sin nx\, dx \right\}$ /2

$= -\frac{1}{\pi n^2}\left[\cos nx\right]_0^{\pi} = \frac{1 - \cos n\pi}{\pi n^2}$ $(n \neq 0)$

$a_0 = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\, dx = \frac{1}{\pi}\left\{ \pi x\Big|_0^{\pi} - \frac{x^2}{2}\Big|_0^{\pi} \right\}$

$= \frac{1}{\pi}\left(\pi^2 - \frac{\pi^2}{2}\right) = \frac{1}{\pi}\frac{\pi^2}{2} = \frac{\pi}{2}$ /1

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$

$= \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\, dx$

$u = \pi - x \Rightarrow u' = -1$

$v' = \sin nx \Rightarrow v = -\frac{\cos nx}{n}$

$b_n = \frac{1}{\pi}\left\{ -\frac{(\pi - x)\cos nx}{n}\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\cos nx\, dx \right\}$

$= \frac{1}{n} - \frac{1}{\pi n^2}\cancel{\left[\sin nx\right]_0^{\pi}} = \frac{1}{n}$ /2

c) $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

$= \frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{1 - \cos n\pi}{n^2}\right)\cos nx$

$+ \sum_{n=1}^{\infty}\frac{\sin nx}{n}$

$= \frac{\pi}{4} + \frac{2}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right)$

$+ \left(\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$

/2

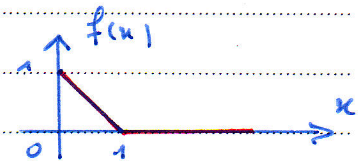
d) $x = 0 \Rightarrow$

$\frac{\pi}{4} + \frac{2}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right) = \frac{f(0^+) + f(0^-)}{2} = \frac{\pi}{2}$

$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\times\frac{\pi}{2} = \frac{\pi^2}{8}$ /2

Ex. 2

a)



/1

b) $F_s(\alpha) = \sqrt{\frac{2}{\pi}}\int_0^1 (1 - x)\sin \alpha x\, dx$

$u = 1 - x \Rightarrow u' = -1$

$v' = \sin \alpha x \Rightarrow v = -\frac{\cos \alpha x}{\alpha}$ /1

$F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left\{ -\frac{(1 - x)\cos \alpha x}{\alpha}\Big|_0^1 - \frac{1}{\alpha}\int_0^1 \cos \alpha x\, dx \right\}$

$= \sqrt{\frac{2}{\pi}}\left\{ \frac{1}{\alpha} - \frac{\sin \alpha x}{\alpha^2}\Big|_0^1 \right\} = \sqrt{\frac{2}{\pi}}\left(\frac{\alpha - \sin\alpha}{\alpha^2}\right)$

c) /3

$\int_0^{\infty}[f(x)]^2 dx = \int_0^1 (1 - x)^2 dx$

$= \int_0^1 (1 - 2x + x^2)\, dx = \left[x - x^2 + \frac{x^3}{3}\right]_0^1$

$= 1 - 1 + \frac{1}{3} = \frac{1}{3}$ /2

$\frac{2}{\pi}\int_0^{\infty}[F_s(\alpha)]^2 d\alpha = \frac{4}{\pi^2}\int_0^{\infty}\left(\frac{\alpha - \sin\alpha}{\alpha^2}\right)^2 d\alpha = \frac{1}{3}$

$\Rightarrow \int_0^{\infty}\left(\frac{\alpha - \sin\alpha}{\alpha^2}\right)^2 = \frac{\pi^2}{12}$ /2