5

Correction of Exam 2 of Maths

Exo1: 1) The matrix A is invertible if $\det(A) \neq 0$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ 5 & -1 \end{vmatrix} + 2 \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} = -27 + 6 = -21 \neq 0 \quad \boxed{1}$$

Then, A is invertible and $A^{-1} = -\frac{1}{21} \begin{pmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{pmatrix}$ $\boxed{2}$

2) The system (S) on the matrix form $AX = B$:

$$AX = B \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix} \quad \boxed{1}$$

3) The solution of the system (S) is $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1} \cdot B$

Then $X = -\frac{1}{21} \begin{pmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{pmatrix} \begin{pmatrix} 6 \\ -4 \\ 27 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$ $\boxed{2}$

Exo2: Let $A = \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$

1) Calculating the characteristic polynomial $P_A(x)$ of A:

$$P_A(x) = |A - xI_3| = \begin{vmatrix} 3-x & -1 & -1 \\ 0 & 2-x & 0 \\ -1 & 1 & 3-x \end{vmatrix}$$

$$= (2-x) \cdot ((3-x)^2 - 1) = (2-x)(4-x)(2-x) \quad \boxed{1}$$

2) The eigenvalues of the matrix A are the roots of the characteristic polynomial $P_A(x)$, Then $\{2, 2, 4\} = \{x_1, x_2, x_3\}$ are eigenvalues. $\boxed{0,5}$

of the eigenvectors $\{v_1, v_2, v_3\}$

For $x_1 = x_2 = 2$,

$$A v_1 = 2 v_1 \iff \begin{pmatrix} 3 & -1 & -1 \\ 0 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix} \iff \begin{cases} x - y - z = 0 \\ 2y = 2y \\ -x + y + z = 0 \end{cases}$$

$\iff x = y + z$, Then, we get $v_{1,2} = \begin{pmatrix} y+z \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow v_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $\boxed{1}$

b) For $x_3 = 4$,

$$A v_3 = 4 v_3 \iff \begin{cases} 3x - y - z = 4x \\ 2y = 4y \\ -x + y + 3z = 4z \end{cases} \iff \begin{cases} y = 0 \\ x = -z \end{cases}$$

$\Rightarrow v_3 = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. $\boxed{0{,}5}$

Then, the eigenvectors are $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

3) The matrix A is diagonalisable, because we have (3) Three eigenvectors. $\boxed{1}$

Exo3: Calculating the indefinite integrals:

a) $\int \frac{\sin(x)\cos(x)}{\sin^2(x)+1} dx$, we can use Bioch's rules,

where $w(x) = \frac{\sin(x)\cos(x)}{\sin^2(x)+1} dx$.

we have $w(-x) = w(x)$, then, we set $t = \sin(x)$ $\boxed{2}$

$$\int \frac{\sin(x)\cos(x)}{\sin^2(x)+1} dx = \int \frac{t}{t^2+1} dt = \frac{1}{2} \ln(t^2+1) + C$$

Finally, $\int \frac{\sin(x)\cos(x)}{\sin^2(x)+1} dx = \frac{1}{2} \ln(\sin^2(x)+1) + C$

$$\int \frac{x}{x^2-2x+2}\,dx = \frac{1}{2}\left[\int \frac{2x-2}{x^2-2x+2}\,dx + \int \frac{2}{x^2-2x+2}\,dx\right]$$

$$= \frac{1}{2}\left[\ln|x^2-2x+2| + 2\int \frac{dx}{(x-1)^2+1}\right]$$

$\boxed{2}$

$$= \frac{1}{2}\left[\ln|x^2-2x+2| + 2\,\mathrm{Arctan}(x-1)\right] + c$$

c) $\displaystyle\int \frac{\ln^3(x)}{x}\,dx = \int \ln^3(x)\cdot\frac{1}{x}\,dx$, we set $t = \ln(x)$

$\Rightarrow dt = \dfrac{dx}{x}$

$\boxed{2}$

Then $\displaystyle\int \frac{\ln^3(x)}{x}\,dx = \int t^3\,dt = \frac{1}{4}t^4 + c = \frac{1}{4}\ln^4(x) + C$

Exo 4:

1) Solving the first ODE: $y' = \dfrac{2x+y-1}{4x+2x+5}$

we have the case $\Delta = 4-4 = 0$, then

$\boxed{1}$

$y' = \dfrac{2x+y-1}{2(2x+y)+5}$ and we tak $z = 2x+y \Rightarrow \dfrac{dz}{dx} = 2 + \dfrac{dy}{dx}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{dz}{dx} - 2$ and the ODE become as follows:

$$\frac{dz}{dx} - 2 = \frac{z-1}{2z+5} \Rightarrow \frac{dz}{dx} = \frac{z-1+4z+10}{2z+5} = \frac{5z+9}{2z+5}$$

$$\Rightarrow \frac{2z+5}{5z+9}\,dz = dx \Leftrightarrow \frac{2}{5}\,\frac{5z+25/2}{5z+9}\,dz = dx$$

$$\Leftrightarrow \frac{2}{5}\left[1 + \frac{7}{2}\,\frac{1}{5z+9}\right]dz = dx \Rightarrow \frac{2}{5}\left[z + \frac{7}{10}\ln(5z+9)\right] = x + c$$

Finaly:

$$\frac{2}{5}\left[(2x+y) + \frac{7}{10}\ln\big(5(2x+y)+9\big)\right] = x + c.$$

$\boxed{1}$

Finding the form of the second ODE.

$$y'' + y' - 2y = (x^2 - 1)e^{-2x}$$
$$= P_2(x)e^{rx}.$$

and the characteristic equation is

$k^2 + k - 2 = 0$ and $\Delta = 9$ and $k_1 = -2$, $k = 1$

The $y_H = c_1 e^{-2x} + c_2 e^{x}$ the homogeneous solution.

$\boxed{1}$

$y_p = ?$:

As $r = -2$ is solution of the characteristic equation

Then, the form of $y_p = (ax^3 + bx^2 + cx + d)e^{-2x}$.

and the general solution

$$y_G = y_H + y_p = c_1 e^{-2x} + c_2 e^{x} + (ax^3 + bx^2 + cx + d)e^{-2x}$$

$\boxed{1}$