Series 1: Numbers and Vectors

Exercise 1.

1. *Show that* $\forall x, y \in \mathbb{R}, x^2 + y^2 = 0 \iff x = y = 0$
2. *Show that* $\forall x, y \in \mathbb{R}, x \neq 2 \wedge y \neq 2 \implies 2x - xy + 2y - 2 \neq 2$.
3. *Show that* n *is prime* $\implies n = 2 \vee n$ *is odd.*

Exercise 2.

1. *Show by contradiction that* $\forall x \in \mathbb{R}^*, \frac{9+x}{2x} \neq \frac{1}{2}$.
2. *Show by induction that* $\forall n \geq 0, \quad 6^n + 9$ *is a multiple of* 5.

Exercise 3. *In each case, give the smallest set to which the following numbers:*

$$\frac{125}{5}, \quad \frac{7}{5}, \quad \frac{21}{12}, \quad \frac{-35}{7}, \quad \frac{14}{21}, \quad \pi, \quad \sqrt{2}$$

Exercise 4. *Which set does the number*

$$\left(\sqrt{n} + \sqrt{\frac{1}{n}}\right)^2$$

belong to, where $n \in \mathbb{N}^*$*?*

Exercise 5.

1. *Show that the square of an odd integer is odd.*
2. *Verify that when* n *is a natural number, the number* $\frac{n(n^2+1)}{2}$ *is also a natural number.*

Exercise 6. *Solve in* $\mathbb{R}$

1)	$(x-2)(3x-1) > 3x - 1$,	5)	$\lvert x+12\rvert = \lvert x^2 - 8\rvert$,
2)	$\dfrac{x^2 - x + 2}{x+1} \leq 1$,	6)	$\lvert 2x - 11\rvert < \lvert x - 5\rvert$,
3)	$2 < \lvert x+1\rvert < 3$,	7)	$\lvert 3x - 4\rvert \leq 1 - x$,
4)	$2\lvert x+2\rvert + \lvert x-5\rvert = 9$,	8)	$\lvert x^2 - 1\rvert \geq 3$.

Exercise 7. .

1. *Show that* $\forall\ a, b \in \mathbb{R}^+; \quad (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$
2. *Solve the equation*

$$\sqrt{41 - x} + \sqrt{41 + x} = 10$$

Exercise 8. *1. Simplify the number:* $A = \dfrac{\sqrt[15]{3^5}\,\sqrt[3]{9}(\sqrt[5]{9})^3}{\sqrt[5]{3}}$

2. *Solve in* $\mathbb{R}$
   1) $x^4 = 16$ *2)* $(x-1)^3+27 = 0,$ *3)* $\sqrt[5]{3x-4} = 2$ *4)* $\sqrt{2-x} = x$

Exercise 9. *Solve in* $\mathbb{R}$

$$1)\quad E(\frac{x-1}{2}) = -2, \qquad 3)\quad E(x) + |x-1| = x,$$
$$2)\quad E(2x) = x-1, \qquad 4)\quad E(\frac{1}{x}) = 3.$$

Exercise 10. *The following families are linearly independent or dependent*

1. $U = (1,0),\quad V = (2,1)$
2. $U = (1,2,3),\quad V = (-1,4,6)$
3. $U = (1,2,-1),\quad V = (1,0,1),\quad W = (0,0,1)$
4. $U = (1,2,-1),\quad V = (1,0,1),\quad W = (-1,2,-3)$
5. $U_1 = (2,-2,-1),\quad U_2 = (1,1,1),\quad U_3 = (1,2,3),\quad U_4 = (2,-1,1).$

Exercise 11. *Show that the vectors* $U_1 = (0,1,1), U_2 = (1,0,1), U_3 = (1,1,0)$ *form a basis of* $\mathbb{R}^3$*. Then find the coordinates of the vector* $V = (1,1,1)$ *with respect to this basis.*

Exercise 12. *1. Show that the vectors* $U = (1,2,1)$*,* $V = (1,9,0)$*,* $W = (3,3,4)$ *form a basis of* $\mathbb{R}^3$*.*

2. *Let* $U_1 = (1,1,4)$*,* $U_2 = (1,3,t)$*,* $U_3 = (1,1,t)$*. For what value of* $t \in \mathbb{R}$ *the set of vectors* $\{U_1,U_2,U_3\}$ *form a basis of* $\mathbb{R}^3$*.*

Exercise 13.

1. *Let* $\mathbf{a} = (3,4)$, $\mathbf{b} = (2,0)$*. Find the vector projection* $\mathbf{proj_b a}$[1]
2. *Let* $\mathbf{a} = (1,\ 2,\ 3)$, $\mathbf{b} = (2,\ -1,\ 4)$*. Find the vector projection* $\mathbf{proj_b a}$*.*

Exercise 14.

1. *Find the rectangular coordinates (in 2D) of the point* $(r,\theta) = (2,\frac{\pi}{4})$ *in polar coordinates.*
2. *Find the rectangular coordinates (in 3D) of the point* $(r,\theta,z) = (2,\frac{4\pi}{3},1)$ *in cylindrical coordinates.*
3. *Find the polar coordinates of the point* $(x,y) = (-1,1)$ *in rectangular coordinates (in 2D).*
4. *Find the cylindrical coordinates of the point* $(x,y,z) = (1,1,3)$ *in rectangular coordinates (in 3D).*

[1] $\mathbf{proj_b a}$: the projection of vector **a** onto vector **b**.