Series 2: Real-Valued Functions

Exercise 1 Calculate the following limits :

1) $\lim_{x\to 0} \ln(x^2+1)\cdot \sin\frac{1}{x^2}$, 2) $\lim_{x\to 0} \frac{\ln(1+x^2)}{\sin^2 x}$, 3) $\lim_{x\to 1} \frac{\sqrt{x}-1}{1-x}$, 4) $\lim_{x\to 0} \frac{1-\cos(2x)}{x^2}$, 5) $\lim_{x\to 0} \frac{\tan x}{\sin x}$.

Exercise 2 Solve in $\mathbb{R}$ the following equations:

1) $\sin x = \frac{1}{2}$, 2) $\cos x = \frac{\sqrt{3}}{2}$, 3) $\tan x = \sqrt{3}$, 4) $\cos(4x) = -2$, 5) $\sin(5x) = \sin\left(\frac{2\pi}{3}+x\right)$,

6) $\cos\left(x+\frac{\pi}{4}\right) = \cos(2x)$, 7) $\sin x\cdot\cos x = \frac{1}{4}$, 8) $\sin\left(2x-\frac{\pi}{3}\right) = \cos\frac{x}{3}$.

Exercise 3 Solve the following inequalities on the interval $[0, 2\pi]$

1) $\sin(x) \geq \frac{1}{2}$, 2) $\cos(x) > \frac{\sqrt{2}}{2}$.

Exercise 4 : 1) Simplify the following expressions:

$a)\ \frac{2\cosh^2 x - \sinh 2x}{x + ln(\cosh x) + ln2}$, $b)\ \tanh(\frac{1}{2}ln3)$.

2) Find to values of x for which:

$$\sinh(2x) = \frac{3}{4}.$$

Exercise 5: Let us define the function f by

$$f(x) = \begin{cases} a(x+1) - b\ln(e-x), & \text{if } x<0,\\ -3, & \text{if } x=0,\\ 2a + b\cos x, & \text{if } x>0. \end{cases}$$

Determine the values of a and b for which the function f is continuous on $\mathbb{R}$.

Exercise 6 Let f be the function defined on $\mathbb{R}$ by:

$$f(x) = 2x^3 - 3x^2 - 1$$

1. Study the variations of the function f.
2. Show that the equation $f(x) = 4$ has a unique solution α on $]2; +\infty[$.

Exercise 7 Show that the equation $x^7 - x^2 + 1 = 0$, has at least one solution in the interval $I = [-2, 0]$.

Exercise 8 Let the function defined by:

$$f(x) = \arccos(2x-1) - \arcsin(3x^2)$$

1. Determine the domain of definition of $f(x)$.
2. Calculate the derivative of $f(x)$.

Exercise 9 Show that for all $x > 0$:

$$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}.$$

Exercise 10 Simplify the following expressions:

1. $\sin(\arccos(x))$,
2. $\cos(\arcsin(x))$,
3. $\sin(2\arcsin(x))$,
4. $\cosh(\operatorname{arcsinh}(x))$ deduce that: $\tanh(\operatorname{arcsinh}(x))$.

Exercise 11 Solve the following equations:

1. $\arcsin x = \arcsin \frac{2}{5} + \arcsin \frac{3}{5}$
2. $\arctan 2x + \arctan 3x = \frac{\pi}{4}$

Exercise 12 1) By using the definitions of hyperbolic functions in terms of exponentials, prove that :

$$\text{for} \quad x \in]-1,1[, \quad \operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$$

2) solve the equation, for $x \in]-1,1[$,

$$\operatorname{artanh}(x) = \ln(3).$$

3) Determine the domain of definition of the function f defined by

$$f(x) = \operatorname{arccosh}\left(\frac{x^2+1}{2x}\right),$$

simplify its expression.

Additional Exercises

Exercise 1 1) Solve in $\mathbb{R}$ the equations:

$a)$ $2\cos^2(x) + 9\cos(x) + 4 = 0$. $b)$ $\cosh(x) = \frac{13}{5}$, $c)$ $\tanh(x) = \frac{3}{5}$, $d)$ $\arccos x = 2\arccos\frac{3}{4}$.

2) Solve the inequalities on the interval $[0, 2\pi]$:

$$a) \quad \sin(x) > -\frac{1}{2}, \quad b) \quad \cos(x) \geq \frac{1}{2}.$$

Exercise 2 For each of the following cases, determine α so that the function is continuous on $\mathbb{R}$:

$$1)\ f(x) = \begin{cases} \frac{x^2-x}{x}, & \text{if } x \neq 0, \\ \alpha, & \text{if } x = 0, \end{cases} \qquad 2)\ f(x) = \begin{cases} \frac{\sqrt{x^2-x+1}-x}{x-1}, & \text{if } x \neq 1, \\ \alpha, & \text{if } x = 1. \end{cases}$$

Exercise 3 Show that the equation

$$e^x - \ln(1+x) - 2 = 0$$

has at least one solution in the interval $I = [0,\ 1]$.

Exercise 4 Show that for all $x \in]-1;1[$: $\arcsin x + \arccos x = \frac{\pi}{2}$.

Exercise 5 Simplify the expression: $\cos(2\arcsin(x))$.