Series 3: Differential Calculus

Exercise 1 Using the l'Hospital's rule, calculate the following limits:

$$\lim_{x\to 0} \frac{2\sin x - \sin(2x)}{x - \sin x}, \qquad \lim_{x\to 0} \frac{e^{4x} - e^{x}}{\tan x}.$$

Exercise 2 Calculate the derivatives of the following functions:

$$e^{\tan x}, \qquad \ln(\tan x), \qquad \arccos(\tan x), \qquad (arctan(x))^4.$$

Exercise 3 Let us consider the two functions defined on $\mathbb{R}$ by

$$f(x) = \begin{cases} x^3 \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases} \qquad g(x) = \ln(1 + |x|).$$

1. Are f and g differentiable at 0?
2. Calculate $f'(x)$.

Exercise 4 Let a, $b \in \mathbb{R}$. We define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} ax + b, & \text{if } x \leq 0, \\ \dfrac{1}{1+x}, & \text{if } x > 0. \end{cases}$$

1. Determine the value of b for which the function f is continuous on $\mathbb{R}$.
2. Determine the values of a and b such that f is differentiable on $\mathbb{R}$, and in this case calculate $f'(0)$.

Exercise 5

1. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{x^2}$.
   Can we apply Rolle's Theorem to the interval $[-1, 1]$?
2. Let $f : [a, b] \to \mathbb{R}$ defined by $f(x) = x^2$.
   Apply the Mean Value Theorem for f and determine c in terms of a and b.

Exercise 6 Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} \dfrac{3 - x^2}{2}, & \text{if } x \leq 1, \\ \dfrac{1}{x}, & \text{if } x > 1. \end{cases}$$

1. Are the hypotheses of the Mean Value Theorem satisfied for the function f on the interval $[0, 2]$?
2. If so, find the values of c in $]0, 2[$.

Additional Exercises

Exercise 1 The functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} - \{-1\} \to \mathbb{R}$ are defined by

$$f(x) = |x| \sin x, \qquad g(x) = |x| + \frac{4}{x+1}$$

Are these functions differentiable at 0?

Exercise 2 Using L'Hôpital's rule, calculate the following limits:

$$\lim_{x\to 0} \frac{e^{2x} - 1}{x}, \qquad \lim_{x\to 1} \frac{1 + \cos(\pi x)}{x^2 - 2x + 1}, \qquad \lim_{x\to +\infty} x^2 e^{-x}.$$

Exercise 3 Verify the Mean Value Theorem for the function

$$f(x) = 2x^2 - 7x + 10$$

on the interval $[2, 5]$.

Exercise 4 Let f be a function defined by

$$f(x) = \begin{cases} \dfrac{1 - \cos(2\pi x)}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Show that there exists $c \in]-1, 1[$ such that $f'(c) = 0$.