Faculty of Science - Badji Mokhtar University - Annaba

Department of Materials of Science

Chapter 01: Ordinary Differential Equations (ODEs)

Course: Mathematics 2

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Chapter 1

Ordinary Differential Equations (ODEs)

Introduction

First-order linear differential equations and second-order differential equations with constant coefficients are among the most important mathematical tools for modeling a wide range of physical, chemical, biological, and economic phenomena.

The study of these equations aims to find functions that satisfy specific relationships between the function and its derivatives, whether of first or second order, allowing us to analyze dynamic systems and understand their behavior accurately.

This chapter includes a selection of solved exercises to illustrate standard methods for solving these equations, strengthen students' analytical skills, and equip them with the tools necessary to tackle real-world problems that can be modeled using first-order linear differential equations or second-order differential equations with constant coefficients.

An Example in Physics

Consider the example of an object sliding on an inclined plane with friction. This involves applying the fundamental principle of Dynamics. We can write, according to Newton's second law,

$$\sum \vec{F}_{\text{ext}} = m\vec{a} = m\frac{d\vec{v}}{dt}.$$

The forces involved are $\vec{P}$, the weight of the object, $\vec{R}$ the reaction force from the support, and $\vec{f}$, the friction force such that

$$\vec{f} = -\lambda\vec{v}, \qquad \lambda \in \mathbb{R}_+^*.$$

Projecting onto the axis of motion, we obtain:

$$mg\sin\alpha - \lambda v = m\frac{dv}{dt},$$

that is,

$$\frac{dv}{dt} + \frac{\lambda}{m}v = g\sin\alpha.$$

We obtain a first-order differential equation with constant coefficients.

Most situations in Physics can be modeled using differential equations. Obviously, the more sophisticated the model, the more complicated the differential equations will be to solve.

1.1 Ordinary Differential Equations

Definition 4.1. An *ordinary differential equation* (ODE) of order n, denoted by ODE, is any relation between the real variable x, an unknown function $x \mapsto y(x)$, and its derivatives $y', y'', \ldots, y^{(n)}$ with $n \geq 1$, of the form:

$$f\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0.$$

Equivalently,

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0,$$

where f is a function that admits infinitely many solutions.

Example

- $y' + xy + \frac{x}{x+1} = 0$ is a differential equation of order 1.
- $x^2y'' + xy' + 2y^4 = 0$ is a differential equation of order 2.
- $xy''' + 2y + xe^x = 0$ is a differential equation of order 3.
- $y''' + 2y^{(8)} = xe^xy'$ is a differential equation of order 8.

Cauchy Problem

Let $y : I \to \mathbb{R}$ such that:

$$\begin{cases} f\left(x, y, y', y^{(2)}, \dots, y^{(n)}\right) = 0, \\ y(x_0) = y_0, \\ y'(x_0) = y'_0, \\ \vdots \\ y^{(n)}(x_0) = y_0^{(n)}, \end{cases}$$

where $x_0, y_0, y'_0, \dots, y_0^{(n)}$ are given values. This problem is called a **Cauchy problem**.

First-Order Differential Equations

A first-order differential equation is of the form:

$$f(x, y, y') = 0,$$

where

$$y'(x) = \frac{dy}{dx}.$$

Separable Variables Equation

Let $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$ be continuous functions. A separable differential equation is of the form:

$$y'(x) = f(x)g(y).$$

Thus,

$$\frac{dy}{dx} = f(x)g(y) \implies g(y)\,dy = f(x)\,dx$$

$$\implies \int g(y)\,dy = \int f(x)\,dx + C, \quad C \in \mathbb{R}.$$

Example 4.2. Solve

$$(x^2 - 1)y' - 2xy = 0 \quad (E).$$

$$(E) \implies \frac{dy}{y} = \frac{2x}{x^2 - 1}dx$$

$$\implies \int \frac{dy}{y} = \int \frac{2x}{x^2 - 1}dx$$

$$\Longrightarrow \ln|y| = \ln|x^2 - 1| + C$$

$$\Longrightarrow y(x) = k(x^2 - 1), \quad k \in \mathbb{R}.$$

1.2 First-Order Linear Differential Equations

1.2.1 General form of a First-Order Linear ODE

Definition 4.3. A first-order linear differential equation is any equation of the form:

$$y'(x) + a(x)y(x) = b(x), \tag{1.1}$$

where $a(x)$ and $b(x)$ are continuous functions defined on $I \subset \mathbb{R}$ or are constants.

Remark

- If $b(x) = 0$, the equation is called **homogeneous**.
- If $b(x) \neq 0$, the equation is called **nonhomogeneous**.

Any general solution of this equation is written as:

$$y(x) = y_h(x) + y_p(x),$$

where:

- y_h is the general solution of the associated homogeneous equation,
- y_p is a particular solution.

1.2.2 Solution of the associated homogeneous equation

Consider:

$$y' + a(x)y = 0.$$

This is a separable equation:

$$\Longrightarrow \frac{dy}{y} = -a(x)\,dx$$

$$\Longrightarrow \ln|y(x)| = -\int a(x)dx + c.$$

$$\Longrightarrow y = e^{-\int a(x)dx + c}$$

Hence,

$$y_h(x) = \lambda e^{-\int a(x)dx}, \quad \lambda \in \mathbb{R}.$$

1.2.3 Particular solution using the method of variation of constants

We look for a particular solution of the form:

$$y_p(x) = \lambda(x) e^{-\int a(x)dx}.$$

Then:

$$y_p'(x) = \lambda'(x) e^{-\int a(x)dx} - \lambda(x) a(x) e^{-\int a(x)dx}.$$

y_p is the particular solution of equation (1.1), i.e.

$$y_p'(x) + a(x) y_p(x) = b(x),$$

$$\lambda'(x) e^{-\int a(x)dx} - \lambda(x) a(x) e^{-\int a(x)dx} + a(x)\lambda(x) e^{-\int a(x)dx} = b(x),$$

$$\lambda'(x) e^{-\int a(x)dx} = b(x).$$

Thus

$$\lambda(x) = \int b(x) e^{\int a(x)dx} dx.$$

After integration, we obtain $\lambda(x)$, and thus the general solution:

$$y_p(x) = \lambda(x) e^{-\int a(x)dx} = e^{-\int a(x)dx} \int b(x) e^{\int a(x)dx} dx.$$

1.2.4 Example

Example 1.2.1 *Solve:*

$$y'(x) - \frac{1}{x} y(x) = \frac{x}{1+x^2}.$$

The homogeneous solution is:

$$y_s(x) = \lambda x.$$

Using variation of constants:

$$y(x) = \lambda(x)x, \quad y'(x) = \lambda'(x)x + \lambda(x).$$

Substitution gives:

$$\lambda'(x) = \frac{1}{1+x^2}.$$

Thus,

$$\lambda(x) = \arctan x + k,$$

and the general solution is:

$$y(x) = x \arctan x + kx.$$

Example 1.2.2 1. *Consider the differential equation*

$$y' - y = e^x \tag{E1}$$

We start by solving the corresponding homogeneous equation

$$y' - y = 0 \tag{Eh1}$$

Solving, we have

$$\frac{dy}{dx} = y \implies \int \frac{dy}{y} = \int dx \implies \ln|y| = x + c, \quad c \in \mathbb{R} \implies y = Ke^x, \quad K = \pm e^c.$$

*Next, we look for a particular solution using the Variation of Constants Method** (VCM)**. We set*

$$y(x) = k(x)e^x \implies y' = k'(x)e^x + k(x)e^x.$$

Substituting into (E1), we get

$$k'(x)e^x + k(x)e^x - k(x)e^x = e^x \implies k'(x) = 1 \implies k(x) = x.$$

Thus, a particular solution is

$$y_p(x) = xe^x.$$

Finally, the general solution of the nonhomogeneous equation is

$$y(x) = Ke^x + xe^x.$$

Theorem 1 (Cauchy problem). *For $x_0 \in I$ and $y_0 \in \mathbb{R}$, the equation (1.1) admits a unique solution of class C^1 on I such that*

$$y(x_0) = y_0.$$

Example 1.2.3 *Consider the differential equation*

$$y' + 2ty = 2t, \quad y(0) = 0 \tag{E2}$$

We start by solving the homogeneous equation

$$y' + 2ty = 0 \tag{Eh2}$$

Solving, we have

$$\frac{dy}{dt} = -2ty \implies \int \frac{dy}{y} = -\int 2t\,dt \implies \ln|y| = -t^2 + c \implies y = Ke^{-t^2}, \quad K = \pm e^c.$$

*Next, we look for a particular solution using the **method of variation of constants (MVC)**. We set*

$$y(t) = k(t)e^{-t^2} \implies y' = k'(t)e^{-t^2} - 2tk(t)e^{-t^2}.$$

Substituting into (E2), we get

$$k'(t)e^{-t^2} - 2tk(t)e^{-t^2} + 2tk(t)e^{-t^2} = 2t \implies k'(t)e^{-t^2} = 2t \implies k'(t) = 2te^{t^2}.$$

Integrating, we find

$$k(t) = \int 2te^{t^2}\,dt = e^{t^2}.$$

Thus, a particular solution is

$$y_p(t) = e^{t^2}e^{-t^2} = 1.$$

Finally, the general solution of the nonhomogeneous equation is

$$y(t) = Ke^{-t^2} + 1.$$

Using the initial condition $y(0) = 0$, we find $K = -1$, so the solution satisfying the initial condition is

$$y(t) = 1 - e^{-t^2}.$$

Example 1.2.4 1. *Consider the differential equation*

$$(E) : y' + \frac{1}{x}y = \ln(x)$$

We first solve the corresponding homogeneous equation

$$(E_h) : y' + \frac{1}{x}y = 0$$

Solving, we have

$$\frac{dy}{dx} = -\frac{y}{x} \implies \int \frac{dy}{y} = -\int \frac{dx}{x} \implies \ln|y| = -\ln|x| + c \implies y = \frac{K}{x}, \quad K = \pm e^c,\ c \in \mathbb{R}.$$

*Next, we look for a particular solution using the **method of variation of constants (MVC)**. We set*

$$y(x) = \frac{k(x)}{x} \implies y' = \frac{k'(x)}{x} - \frac{k(x)}{x^2}.$$

Substituting into (E), we get

$$\frac{k'(x)}{x} - \frac{k(x)}{x^2} + x \cdot \frac{k(x)}{x} = \frac{k'(x)}{x} = \ln(x) \implies k'(x) = x\ln(x).$$

Integrating by parts:

$$k(x) = \int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}.$$

Thus, a particular solution is

$$y_p(x) = \frac{k(x)}{x} = \frac{1}{x}\left(\frac{x^2}{2}\ln(x) - \frac{x^2}{4}\right) = \frac{x}{2}\ln(x) - \frac{x}{4}.$$

Finally, the general solution of the nonhomogeneous equation is

$$y(x) = \frac{K}{x} + \frac{x}{2}\ln(x) - \frac{x}{4}.$$

2. *Consider the differential equation*

$$(E) : y' + 2y = 1$$

We first solve the homogeneous equation

$$(E_h) : y' + 2y = 0$$

Solving, we have

$$\frac{dy}{dx} = -2y \implies \int \frac{dy}{y} = -2\int dx \implies \ln|y| = -2x + c \implies y = Ke^{-2x}, \quad K = \pm e^c,\ c \in \mathbb{R}.$$

*Next, we look for a particular solution using the **method of variation of constants (MVC)**. We set*

$$y(x) = k(x)e^{-2x} \implies y' = k'(x)e^{-2x} - 2k(x)e^{-2x}.$$

Substituting into (E), we get

$$k'(x)e^{-2x} - 2k(x)e^{-2x} + 2k(x)e^{-2x} = k'(x)e^{-2x} = 1 \implies k'(x) = e^{2x}/2.$$

Integrating:

$$k(x) = \frac{1}{2}\int e^{2x}dx = \frac{1}{4}e^{2x}.$$

Thus, a particular solution is

$$y_p(x) = k(x)e^{-2x} = \frac{1}{4}e^{2x}e^{-2x} = \frac{1}{4}.$$

Finally, the general solution of the nonhomogeneous equation is

$$y(x) = Ke^{-2x} + \frac{1}{2}.$$

Example 1.2.5 *1. Consider*

$$xy' - y = x^2 \quad \Longleftrightarrow \quad y' - \frac{1}{x}y = x,$$

a first-order linear differential equation with

$$a(x) = -\frac{1}{x}, \qquad b(x) = x.$$

General solution:

$$y_G = \lambda e^{-\int a(x)\,dx} = \lambda e^{-\int (-\frac{1}{x})dx} = \lambda e^{\int \frac{1}{x}dx} = \lambda|x| = Cx, \quad C \in \mathbb{R}.$$

Particular solution:

Let

$$y_P = C(x)x,$$

where $C(x)$ is a function of x. Then

$$y_P' = C'(x)x + C(x).$$

Substituting into the differential equation:

$$y_P' - \frac{1}{x}y_P = C'(x)x + C(x) - \frac{1}{x}C(x)x = C'(x)x.$$

Thus,

$$C'(x)x = x \quad \Longrightarrow \quad C'(x) = 1.$$

Integrating:

$$C(x) = \int 1\,dx = x.$$

Hence,

$$y_P = C(x)x = x \cdot x = x^2.$$

General solution of the equation:

$$y = y_G + y_P = x^2 + Cx, \quad C \in \mathbb{R}.$$

2. *Consider*

$$y' + 2y = e^{-x},$$

a first-order linear differential equation with

$$a(x) = 2, \qquad b(x) = e^{-x}.$$

General solution:

$$y_G = \lambda e^{-\int a(x)\,dx} = \lambda e^{-\int 2\,dx} = \lambda e^{-2x}, \quad \lambda \in \mathbb{R}.$$

Particular solution:

Let

$$y_P = \lambda(x)e^{-2x},$$

where $\lambda(x)$ is a function of x. Then

$$y_P' = \lambda'(x)e^{-2x} - 2\lambda(x)e^{-2x}.$$

Substituting into the differential equation:

$$y_P' + 2y_P = \lambda'(x)e^{-2x} - 2\lambda(x)e^{-2x} + 2\lambda(x)e^{-2x} = \lambda'(x)e^{-2x}.$$

Thus,

$$\lambda'(x)e^{-2x} = e^{-x} \quad \Longrightarrow \quad \lambda'(x) = e^{x}.$$

Integrating:

$$\lambda(x) = \int e^{x}\,dx = e^{x}.$$

Hence,

$$y_P = \lambda(x)e^{-2x} = e^{x}e^{-2x} = e^{-x}.$$

General solution of the equation:

$$y = y_G + y_P = \lambda e^{-2x} + e^{-x}, \quad \lambda \in \mathbb{R}.$$

1.3 Second-order linear ODEs with constant coefficients

1.3.1 General form of a second-order linear ODE with constant coefficients

A second-order differential equation has the form:

$$f(x,y,y',y'') = 0, \quad y'' = \frac{d^2y}{dx^2}.$$

Example 1.3.1 *The equations* $y''+4y=0$ *and* $2y''+4y'-5y=0$ *are second-order differential equations.*

Definition 1.3.1 *A linear second-order differential equation with constant coefficients is of the form:*

$$ay'' + by' + cy = f(x), \tag{E}$$

where $a,b,c \in \mathbb{R}$ *and* f *is continuous.*

The associated homogeneous equation is:

$$ay'' + by' + cy = 0, \tag{H.E}$$

It is an equation without a second member.

Proposition 1.3.1 *A general solution* y_g *of* (E) *is given by*

$$y_g = y_h + y_p,$$

where y_h *is a solution of* (H.E) *and* y_p *is a particular solution of* (E).

Theorem 2 (**Cauchy Problem**)**.** For $(y_0, y_1) \in \mathbb{R}^2$, the equation (E) admits a unique solution y on I such that

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_1.$$

1.3.2 Solution of the homogeneous equation

Step 1: Solution of the Homogeneous Equation

Let us look for solutions of the homogeneous equation (H.E) of the form

$$y = e^{rx}, \quad r \in \mathbb{R}.$$

Then $\quad y' = re^{rx}, \quad y'' = r^2 e^{rx}.$

Substituting into (H.E), we obtain

$$ar^2 + br + c = 0. \tag{C.E}$$

Equation (C.E) is called the **characteristic equation** associated with (H.E), and its roots r_1, r_2 are called the **characteristic values**. There are three cases depending on the sign of the discriminant $\quad \Delta = b^2 - 4ac.$

1. $\Delta > 0$: The equation (C.E) has two distinct real roots $r_1 \neq r_2$, and the solution of (H.E) is
$$\boxed{y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}.}$$
2. $\Delta = 0$: The equation (C.E) has a double root $r_1 = r_2 = r$, and the solution of (H.E) is
$$\boxed{y(x) = (c_1 x + c_2)e^{rx}.}$$
3. $\Delta < 0$: The equation (C.E) has two complex conjugate roots $r_1 = \alpha + \beta i$, $r_2 = \alpha - \beta i$, and the solution of (H.E) is
$$\boxed{y(x) = e^{\alpha x}\left(c_1 \cos(\beta x) + c_2 \sin(\beta x)\right).}$$

where $c_1, c_2 \in \mathbb{R}$.

Example

1. $y'' - 4y' + 3y = 0$.

   The characteristic equation is

   $$r^2 - 4r + 3 = 0,$$

   with $\Delta = 4$, hence $r_1 = 1$, $r_2 = 3$. The solution is

   $$\boxed{y(x) = c_1 e^x + c_2 e^{3x}, \quad c_1, c_2 \in \mathbb{R}}.$$

2. $y'' + 2y' + y = 0$.

   The characteristic equation is

   $$r^2 + 2r + 1 = 0,$$

   with $\Delta = 0$, hence $r_1 = r_2 = r = -1$. The solution is

   $$\boxed{y(x) = (c_1 x + c_2)e^{-x}, \quad c_1, c_2 \in \mathbb{R}.}$$

3. $y'' + 2y' + 4y = 0$.

   The characteristic equation is

   $$r^2 + 2r + 4 = 0,$$

   with $\Delta = -12$, hence

   $$r_1 = -1 + \sqrt{3}i, \quad r_2 = -1 - \sqrt{3}i.$$

   The solution is

   $$\boxed{y(x) = e^{-x}\left(c_1 \cos(\sqrt{3}x) + c_2 \sin(\sqrt{3}x)\right), \quad c_1, c_2 \in \mathbb{R}.}$$

Method of Solution

The general solution is: $y_g = y_h + y_p$, where:

- y_h is the general solution of the homogeneous equation (H.E):

$$ay'' + by' + cy = 0,$$

- y_p is a particular solution of the full nonhomogeneous equation (E).

Steps to Solve the Equation

1. **Solve the homogeneous equation (find y_h):** Use the characteristic equation

$$ar^2 + br + c = 0.$$

2. **Find a particular solution y_p:** This depends on the form of $f(x)$. Common methods include:
   - Method of variation of constants (a more general method).
   - Method of undetermined coefficients (used when $f(x)$ is a polynomial, exponential, sine, or cosine).

1.3.3 Particular solutions

Step 2: Particular Solution

Finding a particular solution y_p: Method of variation of constants

Let y_1, y_2 be two linearly independent solutions of the homogeneous equation (E.H). We look for a particular solution of (E) in the form

$$y = c_1 y_1 + c_2 y_2,$$

where c_1 and c_2 are functions satisfying

$$\begin{cases} c_1'(x) y_1 + c_2'(x) y_2 = 0, \\ c_1'(x) y_1' + c_2'(x) y_2' = \dfrac{f(x)}{a}. \end{cases}$$

Example 1.3.2 *Solve the differential equation*

$$y'' - 3y' + 2y = e^x. \tag{1.9}$$

1. *The associated homogeneous equation is*

$$y'' - 3y' + 2y = 0.$$

Its characteristic equation is

$$r^2 - 3r + 2 = 0,$$

with roots $r_1 = 1$ and $r_2 = 2$. Hence,

$$y_h = c_1 e^x + c_2 e^{2x}.$$

2. *We seek a particular solution in the form*

$$y_p = c_1(x)e^x + c_2(x)e^{2x},$$

where $c_1(x)$ and $c_2(x)$ satisfy

$$\begin{cases} c_1'(x)e^x + c_2'(x)e^{2x} = 0, \\ c_1'(x)e^x + 2c_2'(x)e^{2x} = e^x. \end{cases}$$

Multiplying the first equation by 2 and subtracting it from the second, we obtain

$$c_1'(x) = -1 \quad \Longrightarrow \quad c_1(x) = -x + c_3.$$

Subtracting the first equation from the second, we obtain

$$c_2'(x) = e^{-x} \quad \Longrightarrow \quad c_2(x) = -e^{-x} + c_4.$$

Thus,

$$y_p = (-x + c_3)e^x + (-e^{-x} + c_4)e^{2x}.$$

Finally, the general solution is

$$y_g = y_h + y_p.$$

Methods for Solving Second-Order Differential Equations with Constant Coefficients

The forms of the particular solutions for different types of right-hand sides are presented in the following table.

Right-hand side $f(x)$	**Roots of the characteristic equation**	**Form of the particular solution** $y_p(x)$
$p_n(x)$ Polynomial of degree n	0 is not a root of $ar^2 + br + c = 0$	$y_p = q_n(x)$
$p_n(x)$ Polynomial of degree n	0 is a root of multiplicity m	$y_p = x^m q_n(x)$
$p_n(x)e^{\alpha x}$	α is not a root	$y_p = q_n(x)e^{\alpha x}$
$p_n(x)e^{\alpha x}$	α is a root of multiplicity m	$y_p = x^m q_n(x)e^{\alpha x}$
$Ae^{\alpha x}$	α is not a root	$y_p = Be^{\alpha x}$
$Ae^{\alpha x}$	α is a simple root	$y_p = Bxe^{\alpha x}$
$Ae^{\alpha x}$	α is a double root	$y_p = Bx^2 e^{\alpha x}$
$p_n(x)\cos\beta x + q_n'(x)\sin\beta x$	$\pm i\beta$ are not roots	$y_p = p_n(x)\cos\beta x + q_n'(x)\sin\beta x$
$p_n(x)\cos\beta x + q_n'(x)\sin\beta x$	$\pm i\beta$ are roots of multiplicity m	$y_p = x^m\Big(p_n(x)\cos\beta x + q_n'(x)\sin\beta x\Big)$
$e^{\alpha x}(p_n(x)\cos\beta x + q_n'(x)\sin\beta x)$	$\alpha \pm i\beta$ are not roots	$y_p = e^{\alpha x}(p_n(x)\cos\beta x + q_n'(x)\sin\beta x)$
$e^{\alpha x}(p_n(x)\cos\beta x + q_n'(x)\sin\beta x)$	$\alpha \pm i\beta$ are roots of multiplicity m	$y_p = x^m e^{\alpha x}(p_n(x)\cos\beta x + q_n'(x)\sin\beta x)$

1.3.4 Example

Example

1. $y'' + 2y' - 3y = -6x^2 - x + 7$
2. $y'' + 5y' + 6y = 2e^{2x}$
3. $y'' + 5y' + 6y = 2e^{-3x}$

Solution

1. Solve the homogeneous equation (HE)

$$y'' + 2y' - 3y = 0$$

Characteristic equation:

$$r^2 + 2r - 3 = 0, \qquad \Delta = 16$$

Roots:

$$r_1 = -3, \quad r_2 = 1$$

Hence, the general solution of the homogeneous equation is

$$y_h = \lambda_1 e^{-3x} + \lambda_2 e^x, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Solve the nonhomogeneous equation (NHE)

$$y'' + 2y' - 3y = -6x^2 - x + 7$$

Here,

$$f(x) = p_2(x) = -6x^2 - x + 7.$$

Since 0 is not a root of the characteristic equation $r^2 + 2r - 3 = 0$, the particular solution is of the form

$$y_p = q_2(x) = ax^2 + bx + c.$$

Compute derivatives:

$$y_p' = 2ax + b, \qquad y_p'' = 2a.$$

Substitute into the NHE:

$$2a + 2(2ax + b) - 3(ax^2 + bx + c) = -6x^2 - x + 7$$

Simplify:

$$-3ax^2 + (4a - 3b)x + (2a + 2b - 3c) = -6x^2 - x + 7$$

Equate coefficients:

$$\begin{cases} -3a = -6, \\ 4a - 3b = -1, \\ 2a + 2b - 3c = 7 \end{cases} \quad \Rightarrow \quad a = 2,\ b = 3,\ c = 1$$

Thus, the particular solution is

$$y_p = 2x^2 + 3x + 1.$$

General solution:

$$y = y_h + y_p = 2x^2 + 3x + 1 + \lambda_1 e^{-3x} + \lambda_2 e^{x}, \qquad \lambda_1, \lambda_2 \in \mathbb{R}.$$

2. **Solve the homogeneous equation** (**HE**)

$$y'' + 5y' + 6y = 0$$

Characteristic equation:

$$r^2 + 5r + 6 = 0, \quad \Delta = 1$$

Roots:

$$r_1 = -3, \quad r_2 = -2$$

Hence, the general solution of the homogeneous equation is

$$y_h = \lambda_1 e^{-3x} + \lambda_2 e^{-2x}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Nonhomogeneous equation (NHE)

$$y'' + 5y' + 6y = 2e^{2x}$$

Here,

$$f(x) = 2e^{2x} = Ae^{\alpha x}, \quad \alpha = 2$$

Since $\alpha = 2$ is not a root of the characteristic equation $r^2 + 5r + 6 = 0$, the particular solution is of the form

$$y_p = Be^{2x}.$$

Compute derivatives:

$$y_p' = 2Be^{2x}, \qquad y_p'' = 4Be^{2x}.$$

Substitute into the NHE:

$$y_p'' + 5y_p' + 6y_p = 4Be^{2x} + 10Be^{2x} + 6Be^{2x} = 20Be^{2x}.$$

Equate to the RHS:

$$20Be^{2x} = 2e^{2x} \quad \Rightarrow \quad B = \frac{1}{10}.$$

Thus, the particular solution is

$$y_p = \frac{1}{10}e^{2x}.$$

General solution:

$$y = y_h + y_p = \frac{1}{10}e^{2x} + \lambda_1 e^{-3x} + \lambda_2 e^{-2x}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

3. Solve the homogeneous equation (HE)

$$y'' + 5y' + 6y = 0$$

Characteristic equation:

$$r^2 + 5r + 6 = 0$$

Roots:

$$r_1 = -3, \quad r_2 = -2$$

Hence, the general solution of the homogeneous equation is

$$y_h = \lambda_1 e^{-3x} + \lambda_2 e^{-2x}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$

Nonhomogeneous equation (NHE)

$$y'' + 5y' + 6y = 2e^{-3x}$$

Here,

$$f(x) = 2e^{-3x} = Ae^{\alpha x}, \quad \alpha = -3$$

Since $\alpha = -3$ is a **simple root** of the characteristic equation $r^2 + 5r + 6 = 0$, the particular solution is of the form

$$y_p = Bxe^{-3x}.$$

Compute derivatives:

$$y_p' = Be^{-3x} - 3Bxe^{-3x} = (B - 3Bx)e^{-3x}, \quad y_p'' = -6Be^{-3x} + 9Bxe^{-3x} = (-6B + 9Bx)e^{-3x}.$$

Substitute into the NHE:

$$y_p'' + 5y_p' + 6y_p = (-6B + 9Bx + 5B - 15Bx + 6Bx)e^{-3x} = (-B)e^{-3x}.$$

Equate to the RHS:

$$-Be^{-3x} = 2e^{-3x} \quad \Rightarrow \quad B = -2.$$

Thus, the particular solution is

$$y_p = -2xe^{-3x}.$$

General solution:

$$y = y_h + y_p = -2xe^{-3x} + \lambda_1 e^{-3x} + \lambda_2 e^{-2x}, \quad \lambda_1, \lambda_2 \in \mathbb{R}.$$