Methods for Solving Second-Order Differential Equations with Constant Coefficients

The forms of the particular solutions for different types of right-hand sides are presented in the following table: (used when f (x) is a polynomial, exponential, sine, or cosine)

Right-hand side $f(x)$	Root of the characteristic equation	Form of the particular solution
$p_n(x)$ Polynomial of degree n	Zero is not a root of the characteristic equation: $ar^2+br+c=0$	$y_p = q_n(x)$
	Zero is a root of multiplicity m of the characteristic equation $ar^2+br+c=0$	$y_p = x^m q_n(x)$
$p_n(x)e^{\alpha x}$	α is not a root of the characteristic equation $ar^2+br+c=0$	$y_p = q_n(x)e^{\alpha x}$
	α is a root of multiplicity m of the characteristic equation $ar^2+br+c=0$	$y_p = x^m q_n(x)e^{\alpha x}$
$Ae^{\alpha x}$	α is not a root of the characteristic equation $ar^2+br+c=0$	$y_p = Be^{\alpha x}$
	α is a simple root of the characteristic equation $ar^2+br+c=0$	$y_p = Bxe^{\alpha x}$
	α is a double root of the characteristic equation $ar^2+br+c=0$	$y_p = Bx^2e^{\alpha x}$
$p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x$	$\pm i\beta$ are not roots of the characteristic equation $ar^2+br+c=0$	$y_p = p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x$
	$\pm i\beta$ are roots of multiplicity m of the characteristic equation $ar^2+br+c=0$	$y_p = x^m\big(p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x\big)$
$e^{\alpha x}\big(p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x\big)$	$\alpha \pm i\beta$ are not roots of the characteristic equation $ar^2+br+c=0$	$y_p = e^{\alpha x}\big(p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x\big)$
	$\alpha \pm i\beta$ are roots of multiplicity m of the characteristic equation $ar^2+br+c=0$	$y_p = x^m e^{\alpha x}\big(p_n(x)\cos\beta x + q_{n'}(x)\sin\beta x\big)$

Example

1.
$$y'' + 2y' - 3y = -6x^2 - x + 7$$

2.
$$y'' + 5y' + 6y = 2e^{2x}$$

3.
$$y'' + 5y' + 6y = 2e^{-3x}$$

Method of variation of constants (a more general method).

We use this method to solve a second-order linear non-homogeneous differential equation of the form:

$$ay'' + by' + cy = f(x)$$

Solve the homogeneous equation

$$ay'' + by' + cy = 0$$

Find two linearly independent solutions $y_1(x)$ and $y_2(x)$. These form a fundamental set of solutions. The general solution of the homogeneous equation is:

$$y_h = \lambda_1 y_1(x) + \lambda_2 y_2(x)$$

Look for a particular solution of the non-homogeneous equation We assume a solution of the form:

$$y_p = \lambda_1(x) y_1(x) + \lambda_2(x) y_2(x)$$

where $\lambda_1(x)$ and $\lambda_2(x)$ are functions to be determined (variation of constants).

satisfying:

$$\begin{cases} \lambda_1'(x) y_1(x) + \lambda_2'(x) y_2(x) = 0 \\ \lambda_1'(x) y_1'(x) + \lambda_2'(x) y_2'(x) = f(x) \end{cases}$$

Solve for $\lambda_1'(x)$ and $\lambda_2'(x)$ Solve the system of two linear equations

Then integrate to find $\lambda_1(x)$ and $\lambda_2(x)$, giving the particular solution y_p.

Example

Solve the following equations:

1. $$y'' + 3y' - 4y = e^x$$

2. $$y'' - y' - 2y = e^{3x}$$