Series 2

Exercice 1 *We consider the matrices*

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \ B = \begin{pmatrix} -2 & 0 \\ 7 & 1 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 0 & 1 \\ 1 & 7 & 6 \end{pmatrix}, \ D = \begin{pmatrix} 4 & 5 \\ 1 & 2 \\ 0 & 4 \end{pmatrix}$$

Calculate : $AB, \ BA, \ CD, \ 2A + B, \ A - 4B, \ C^2, \ (AB)^t.$

Exercice 2 *Let* $A = \begin{pmatrix} x & 8 \\ 0 & 2x \end{pmatrix}, \ B = \begin{pmatrix} y & 7 \\ -1 & 3y \end{pmatrix}$

Find x *and* y *such that* $2A - 4B = \begin{pmatrix} -5 & -18 \\ 4 & -16 \end{pmatrix}.$

Exercice 3 *Calculate the determinant of the matrices :*

$$A = \begin{pmatrix} 7 & 11 \\ -8 & 4 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 & 6 \\ 3 & 4 & 15 \\ 5 & 6 & 21 \end{pmatrix}, \ C = \begin{pmatrix} -1 & 1 & 2 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \ D = \begin{pmatrix} 2 & 0 & 7 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & 0 & 4 & 0 \\ 0 & 7 & 0 & 0 \end{pmatrix}.$$

Exercice 4 *Let* $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

1. *Find* $a, b \in \mathbb{R}$ *such that* $A^2 = aI_3 + bA.$
2. *Deduce that* A *is invertible and give* $A^{-1}.$

Exercice 5 *Let the following matrices :*

$$M = \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix}, \quad N = \begin{pmatrix} 1 & 0 & -1 \\ -2 & 3 & 4 \\ 0 & 1 & 1 \end{pmatrix}$$

M, N are they invertible? If yes give $M^{-1}, \ N^{-1}.$

Exercice 6 *Let* $\alpha \in \mathbb{R}$, *and* A_α *be a matrix in* $M_3(\mathbb{R})$ *defined by :*

$$A_\alpha = \begin{pmatrix} 4 & -1 & 3 \\ 2 & 2 & 1 \\ \alpha & 1 & \alpha - 2 \end{pmatrix}$$

1. *Calculate the determinant of* $A_\alpha.$
2. *Determine the values of* α *for the matrix* A_α *to be invertible.*

Exercice 7   1. *Find the matrix associated of the linear application* f *defined by.*

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$f(x, y, z) = (5x + y + 7z, -10y + 2z, 7x + y - 12z)$$

2. *Find the linear application associated of the matrix* $A = \begin{pmatrix} 2 & 0 & \sqrt{2} \\ 5 & 0.2 & 1 \\ 1 & 5 & 6 \end{pmatrix}.$

Exercice 8 *Let*

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$f(x, y, z) = (x, z, -y)$$

the linear transformation.

1. *Show that the matrix associated with f in the canonical basis is the matrix A defined by*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

2. *Calculate $A^3 - A^2 + A - I_3$.*
3. *deduce A is invertible and give A^{-1}.*
4. *Find A^{-1} using the cofactor matrix methode.*

Exercice 9 *(Additional Exercise)*
   I) Consider the matrix

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

Let $B = A + 3I$

1. *Express B^2 in termes of B.*
2. *Deduce A^2 in termes of A.*
3. *Is the matrix A invertible ?*
   II) Let A be the matrix defined by

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -3 & 2 \\ -1 & 2 & 0 \end{pmatrix}$$

4. *Is the matrix A invertible ? If yes, determine its inverse A^{-1}.*
5. *Calculate A^2, then find α, $\beta \in \mathbb{R}$ such that : $A^2 = \alpha A + \beta I_3$.*
6. *Deduce A^{-1}.*

Exercice 10 *Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}$.*

1. *Calculate A^n and B^n for all $n \in \mathbb{N}^*$.*
2. *Show that $AB = BA$.*
3. *Calculate $(A + B)^n$ by Newton's binomial formula $(A + B)^n = \sum_{p=0}^{n} C_n^p \, A^{n-p} \, B^p$.*

Exercice 11 *Calculate the rank of the following matrices :*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}, \ C = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 3 & 4 & 5 & 2 \end{pmatrix}, \ D = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -2 & -3 & 0 & -5 \\ 4 & 9 & 6 & 7 \\ 1 & -1 & -5 & 5 \end{pmatrix}.$$

Exercice 12 *Solve the following linear systems (by Cramer's method)*

$$(1) \begin{cases} x + y + 2z = 3 \\ x + 2y + z = 1 \\ 2x + y - z = 0 \end{cases}, \qquad (2) \begin{cases} x + 2z = 1 \\ -y + z = -7 \\ x - 2y = 1 \end{cases}$$

Exercice 13 *Solve the following linear systems (by the inverse matrix method)*

$$(1) \begin{cases} 5x + 7y - 3z = 16 \\ 3x - 2y + 4z = -7 \\ x + y - z = 6 \end{cases}, \qquad (2) \begin{cases} x + 2y = z + 1 \\ -x + 2y = -3 \\ -x + y = -2 - 2z \end{cases}$$

Exercice 14 *Solve the following linear systems (by the Gauss's method)*

$$(1) \begin{cases} -y + 2z - 13t = 5 \\ x - 2y + 3z + 17t = 4 \\ -x + 3y - 3z - 20t = -1 \end{cases}, \qquad (2) \begin{cases} x + y + t = 1 \\ x + 2y + 2z + 4t = 2 \\ x + 3y + 3z + 7t = 4 \end{cases}$$

Exercice 15 *Let m be a given real parameter. Consider the following linear system :*

$$(S_m) \begin{cases} x + y - z = m - 1 \\ x + (m - 1)y + z = m - 1 \\ -x + y + (m - 1)z = 1 - m \end{cases}$$

1. *Find the values of the parameter m such that the system (S_m) is a Cramer's system.*
2. *Let $m = 2$. Solve the system (S_2) using Cramer's method.*

Exercice 16 *Let the matrix* $A = \begin{pmatrix} 1 & -2 & 4 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$.

1. *Show that A is invertible and give A^{-1}.*
2. *Deduce the solution of the following system*

$$\begin{cases} -x + y - z = 10 \\ x + 2y + z = -4 \\ x - 2y + 4z = 6 \end{cases}$$