People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

Badji Mokhtar Annaba University
Department of Material Sciences
First Year Common Core: Material Sciences

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Solutions to the Mathematics 2 Exam

Solution 1 (6 pts)
Consider

$$A_\alpha = \begin{pmatrix} 1 & 2 & 1 \\ \alpha & 2\alpha & 2\alpha \\ 0 & 4\alpha & 4 \end{pmatrix}$$

1. Invertibility of A_α and rank

   Computation of the determinant

$$\det(A_\alpha) = \begin{vmatrix} 1 & 2 & 1 \\ \alpha & 2\alpha & 2\alpha \\ 0 & 4\alpha & 4 \end{vmatrix}$$

Expanding along the first row:

$$\det(A_\alpha) = 1\begin{vmatrix} 2\alpha & 2\alpha \\ 4\alpha & 4 \end{vmatrix} - 2\begin{vmatrix} \alpha & 2\alpha \\ 0 & 4 \end{vmatrix} + 1\begin{vmatrix} \alpha & 2\alpha \\ 0 & 4\alpha \end{vmatrix}$$

$$= 1(8\alpha - 8\alpha^2) - 2(4\alpha) + 4\alpha^2$$

$$= 8\alpha - 8\alpha^2 - 8\alpha + 4\alpha^2$$

$$\boxed{\det(A_\alpha) = -4\alpha^2} \qquad (1pts)$$

Hence,

$$A_\alpha \text{ is invertible} \iff \det(A_\alpha) \neq 0$$

$$\boxed{A_\alpha \text{ is invertible} \iff \alpha \neq 0 \iff \alpha \in \mathbb{R}^*} \qquad (0,5pt)$$

Rank of A_α

- If $\alpha \neq 0$, then $\det(A_\alpha) \neq 0$, therefore

$$\boxed{\text{rank}(A_\alpha) = 3} \qquad (0,25pt)$$

- If $\alpha = 0$,

$$A_0 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

  The two nonzero rows are linearly independent, hence

$$\boxed{\text{rank}(A_0) = 2} \qquad (0,25pt)$$

2. Study of the system (S_α)

   (a) Matrix form of the system
   The system is

$$\begin{cases} x + 2y + z = -4 \\ \alpha x + 2\alpha y + 2\alpha z = 0 \\ 4\alpha y + 4z = 12 \end{cases}$$

   Let

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad B = \begin{pmatrix} -4 \\ 0 \\ 12 \end{pmatrix}$$

   Then

$$\boxed{A_\alpha X = B}$$

   that is,

$$\boxed{\begin{pmatrix} 1 & 2 & 1 \\ \alpha & 2\alpha & 2\alpha \\ 0 & 4\alpha & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 12 \end{pmatrix}} \qquad (0,5pt)$$

   (b) Solving the system for $\alpha = 1$ using Cramer's rule
   For $\alpha = 1$,

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 0 & 4 & 4 \end{pmatrix}$$

   and

$$B = \begin{pmatrix} -4 \\ 0 \\ 12 \end{pmatrix}$$

$$det(A_1) = -4(1) = -4 \neq 0 \qquad (0.25pt) \implies A \text{ is invertible.}$$

   Therefore, the system has a unique solution.

$$A_1 = \begin{pmatrix} -4 & 2 & 1 \\ 0 & 2 & 2 \\ 12 & 4 & 4 \end{pmatrix}, \quad (0.25pt) \quad \det(A_1) = 24, \quad (0.5pt) \quad x = \frac{\det(A_1)}{\det(A)} = \frac{24}{-4} = -6. \qquad (0.25pt)$$

$$A_2 = \begin{pmatrix} 1 & -4 & 1 \\ 1 & 0 & 2 \\ 0 & 12 & 4 \end{pmatrix}, \quad \det(A_2) = 4, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{4}{-4} = -1. \qquad (1pts)$$

$$A_3 = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 2 & 0 \\ 0 & 4 & 12 \end{pmatrix}, \quad \det(A_3) = -16, \quad z = \frac{\det(A_3)}{\det(A)} = \frac{-16}{-4} = 4. \qquad (1pts)$$

   Then, $S = \{(x, y, z) = (-6, -1, 4)\}$, is a unique solution of system (S_1). (0.25 pt)

Solution 2 (2 pts)
We consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & -1 \\ x & y & 3 \end{pmatrix}, \qquad C = \begin{pmatrix} 3 & 4 & 5 \\ 0 & -1 & 4 \end{pmatrix}.$$

i. Determine the real numbers x and y such that

$$A \times B = C.$$

$$\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & -1 \\ x & y & 3 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 0 & -1 & 4 \end{pmatrix}.$$

$$\begin{pmatrix} 1+2x & 2+2y & -1+6 \\ x-1 & y-2 & 1+3 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 0 & -1 & 4 \end{pmatrix}.(1.25pt)$$

$$\begin{pmatrix} 1+2x & 2+2y & 5 \\ x-1 & y-2 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 0 & -1 & 4 \end{pmatrix}. \qquad (0,25pt)$$

Since $AB = C$, we identify the corresponding entries:

$$\begin{cases} 1+2x = 3 \\ 2+2y = 4 \\ x-1 = 0 \\ y-2 = -1 \end{cases}$$

From

$$1 + 2x = 3$$

we obtain

$$\boxed{x = 1} \qquad (0.25pt)$$

From

$$2 + 2y = 4$$

we obtain

$$\boxed{y = 1} \qquad (0.25pt)$$

Therefore,

$$\boxed{x = 1 \quad \text{and} \quad y = 1}$$

Solution 3 (5 pts)
Let

$$f(x) = \frac{x \ln(\cos x)}{1 + x\sqrt{1+x} - e^{\sin x}}$$

i. Calculate the limited developments of order 3 at 0 of

$$x \ln(\cos x) \quad \text{and} \quad 1 + x\sqrt{1+x} - e^{\sin x}.$$

Development of $x \ln(\cos x)$
We use the classical expansion:

$$\cos x = 1 - \frac{x^2}{2} + o(x^2) \qquad (0.5pt)$$

Hence

$$\ln(\cos x) = \ln\left(1 - \frac{x^2}{2} + o(x^2)\right)$$

Using

$$\ln(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} + o(u^3) \qquad (0.5pt)$$

with

$$u = -\frac{x^2}{2} + o(x^2), \qquad (0.25pt)$$

we obtain

$$\ln(\cos x) = -\frac{x^2}{2} + o(x^2)$$

Multiplying by x:

$$x \ln(\cos x) = -\frac{x^3}{2} + o(x^3) \qquad (0.5pt)$$

Therefore,

$$\boxed{x \ln(\cos x) = -\frac{x^3}{2} + o(x^3)}$$

Development of $1 + x\sqrt{1+x} - e^{\sin x}$
First,

$$\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + o(x^3) \qquad (0.5pt)$$

Thus,

$$x\sqrt{1+x} = x + \frac{x^2}{2} - \frac{x^3}{8} + o(x^3)$$

Also,

$$\sin x = x - \frac{x^3}{6} + o(x^3) \qquad (0.5pt)$$

Using

$$e^u = 1 + u + \frac{u^2}{2} + \frac{u^3}{6} + o(u^3), \qquad (0.5pt)$$

with

$$u = x - \frac{x^3}{6} + o(x^3),$$

we get

$$e^{\sin x} = 1 + x + \frac{x^2}{2} + o(x^3) \qquad (0.5pt)$$

Hence

$$1 + x\sqrt{1+x} - e^{\sin x}$$
$$= 1 + \left(x + \frac{x^2}{2} - \frac{x^3}{8}\right) - \left(1 + x + \frac{x^2}{2}\right) + o(x^3)$$
$$= -\frac{x^3}{8} + o(x^3) \qquad (0.75pt)$$

Therefore,

$$\boxed{1 + x\sqrt{1+x} - e^{\sin x} = -\frac{x^3}{8} + o(x^3)}$$

ii. Deduce

$$\lim_{x \to 0} f(x)$$

Using the previous developments,

$$f(x) = \frac{-\frac{x^3}{2} + o(x^3)}{-\frac{x^3}{8} + o(x^3)}$$

Thus,

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{-\frac{x^3}{2} + o(x^3)}{-\frac{x^3}{8} + o(x^3)} = \frac{\frac{-1}{2}}{\frac{-1}{8}} = 4 \qquad (0.5pt)$$

Therefore,

$$\boxed{\lim_{x \to 0} f(x) = 4}$$

Solution 4 (5 pts)

Solve the following differential equations: 1)

$$y' - \frac{x}{1+x^2}y = 0$$

$$y' = \frac{x}{1+x^2}y \qquad (0.5pt)$$

We write

$$\frac{dy}{dx} = \frac{x}{1+x^2}y$$

$$\frac{dy}{y} = \frac{x}{1+x^2}dx \qquad (0.5pt)$$

Integrating both sides:

$$\int \frac{1}{y}\,dy = \int \frac{x}{1+x^2}\,dx$$

$$\int \frac{1}{y}\,dy = \frac{1}{2}\int \frac{2x}{1+x^2}\,dx \qquad (0.5pt)$$

$$\ln|y| = \frac{1}{2}\ln(1+x^2) + C$$

Hence,

$$y = Ce^{\frac{1}{2}\ln(1+x^2)}$$

$$y = Ce^{\ln\sqrt{1+x^2}}$$

we obtain

$$\boxed{y(x) = C\sqrt{1+x^2}} \qquad (0.5pt)$$

where $C \in \mathbb{R}$.

2)

$$y'' - 2y' + y = e^{-x}$$

General solution $\quad y_G = y_h + y_p$

Step 1: Solve the associated homogeneous equation

$$y'' - 2y' + y = 0$$

The characteristic equation is

$$r^2 - 2r + 1 = 0 \qquad (0.5pt)$$

Thus, the double root is

$$r = 1$$

Therefore, the general solution of the homogeneous equation is

$$y_h(x) = (C_1 + C_2x)e^x \qquad (0.5pt)$$

Step 2: Find a particular solution

The right-hand side is

$$e^{-x}$$

We look for a particular solution of the form

$$y_p = Ae^{-x} \qquad \text{‘because -1 is not a root of the characteristic equation} \qquad (0.5pt)$$

Then

$$y_p' = -Ae^{-x}, \qquad y_p'' = Ae^{-x}$$

Substituting into the equation:

$$Ae^{-x} - 2(-Ae^{-x}) + Ae^{-x} = e^{-x}$$

$$4Ae^{-x} = e^{-x}$$

Hence,

$$4A = 1$$

$$A = \frac{1}{4} \qquad (0.5pt)$$

Thus,

$$y_p(x) = \frac{1}{4}e^{-x} \qquad (0.5pt)$$

General solution

$$\boxed{y_G(x) = y_h + y_p = (C_1 + C_2 x)e^{x} + \frac{1}{4}e^{-x} \qquad (0.5pt)}$$

where $C_1, C_2 \in \mathbb{R}$.

Solution 5 (2 pts)
Let

$$f(x,y) = \frac{x^2 + xy}{x^2 + y^2}$$

1) Calculate the partial derivatives

$$\frac{\partial f}{\partial x} \quad \text{and} \quad \frac{\partial f}{\partial y}.$$

Partial derivative with respect to x

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2 + xy}{x^2 + y^2}\right) = \frac{(2x + y)(x^2 + y^2) - (x^2 + xy)(2x)}{(x^2 + y^2)^2} \qquad (0.75pt)$$

Thus,

$$\frac{\partial f(x,y)}{\partial x} = \frac{2x^3 + x^2y + 2xy^2 + y^3 - 2x^3 - 2x^2y}{(x^2 + y^2)^2}$$

$$\boxed{\frac{\partial f(x,y)}{\partial x} = \frac{-x^2y + 2xy^2 + y^3}{(x^2 + y^2)^2}} \qquad (0.25pt)$$

Partial derivative with respect to y

$$\frac{\partial f(x,y)}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2 + xy}{x^2 + y^2}\right) = \frac{x(x^2 + y^2) - (x^2 + xy)(2y)}{(x^2 + y^2)^2} \qquad (0.75pt)$$

Thus,

$$\frac{\partial f(x,y)}{\partial y} = \frac{x^3 + xy^2 - 2x^2y - 2xy^2}{(x^2 + y^2)^2}$$

$$\boxed{\frac{\partial f(x,y)}{\partial y} = \frac{x^3 - 2x^2y - xy^2}{(x^2 + y^2)^2}} \qquad (0.25pt)$$