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NUMBERS AND VECTORS



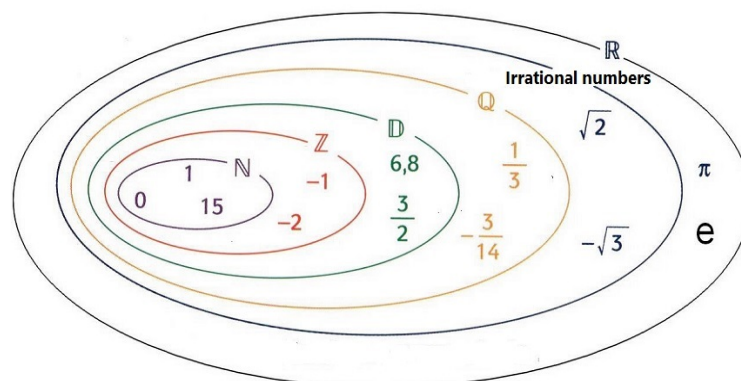
1.1 Real numbers

Definition 1.1.1 The real numbers denoted by $\mathbb{R}$, form the set of all numbers can be represented as follows:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$: The set of all **natural** numbers.
- $\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, \dots\}$: The set of all **integers**.
- $\mathbb{D} = \{\frac{a}{10^n}, \quad a \in \mathbb{Z}, n \in \mathbb{N}\}$: The set of all **decimal** numbers.
- $\mathbb{Q} = \{\frac{p}{q}, \quad p \in \mathbb{Z}, q \in \mathbb{N}^*\}$: The set of all **rational** numbers.
- **Irrational** numbers, noted by $\mathbb{R}/\mathbb{Q} = \{\pi, \sqrt{2}, \sqrt{3}, e, \dots\}$.

So $\mathbb{R} = \mathbb{Q} \cup (\text{Irrational numbers})$.

We can write: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R}$.



Example 1.1.1 *In each case, give the smallest set to which the following numbers:*

- $\frac{125}{5} = 5 \in \mathbb{N}$.
- $\frac{7}{5} = 1,4 = \frac{14}{10} \in \mathbb{D}$.
- $\frac{21}{12} = 1,75 = \frac{175}{100} \in \mathbb{D}$.
- $\frac{-35}{7} = -5 \in \mathbb{Z}$.
- $\frac{14}{21} = 0,666... \in \mathbb{Q}$.
- $\pi \in \mathbb{R}/\mathbb{Q}$ (Irrational numbers).

1.2 Absolute value

Definition 1.2.1 *The absolute value of a real numbers x , denoted by $|x|$, is the distance between x and 0 by*

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

Example 1.2.1 • $|5| = 5, \quad |-3| = 3, \quad |0| = 0$.

Proposition 1.2.1 .

1. $|x| \geq 0, \quad |-x| = x, \quad |x| > 0 \Leftrightarrow x \neq 0$.
2. $\sqrt{x^2} = |x|$.
3. $|x| \leq a \Leftrightarrow -a \leq x \leq a$.
4. $|x| \geq a \Leftrightarrow x \geq a \quad \text{or} \quad x \leq -a$.
5. $|xy| = |x||y|$.
6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
7. $|x + y| \leq |x| + |y|$.
8. $||x| - |y|| \leq |x - y|$.

.

Example 1.2.2 Rewrite $f(x) = |3 - 2x|$ without using the absolute value sign. We have

$$3 - 2x = 0 \Leftrightarrow 3 = 2x \Leftrightarrow x = \frac{3}{2}.$$

So,

$$f(x) = |3 - 2x| = \begin{cases} -(3 - 2x), & \text{if } x \geq \frac{3}{2}, \\ 3 - 2x, & \text{if } x < \frac{3}{2}. \end{cases}$$

or equivalently,

$$f(x) = \begin{cases} -3 + 2x, & \text{if } x \geq \frac{3}{2}, \\ 3 - 2x, & \text{if } x < \frac{3}{2}. \end{cases}$$

Example 1.2.3 Solve in $\mathbb{R}$: $|x - 2| = 3$.

We have $|x - 2| = 3 \Leftrightarrow x - 2 = 3$ or $-x + 2 = 3 \Leftrightarrow x = 5$ or $x = -1$.

So $S = \{-1, 5\}$.

Example 1.2.4 Solve in $\mathbb{R}$: $3 < |x + 2| < 4$.

We have $3 < |x + 2| < 4 \Leftrightarrow 3 < x + 2 < 4$ or $3 < -x - 2 < 4$

$\Leftrightarrow 1 < x < 2$ or $-6 < x < -5 \Leftrightarrow x \in]1; 2[$ or $x \in]-6; -5[$.

So $S =]-6; -5[\cup]1; 2[$.

Example 1.2.5 Solve in $\mathbb{R}$: $|2x + 3| \leq 5$.

We have $|2x + 3| \leq 5 \Leftrightarrow -5 \leq 2x + 3 \leq 5$

$\Leftrightarrow -8 \leq 2x \leq 2 \Leftrightarrow -4 \leq x \leq 1 \Leftrightarrow x \in [-4; 1]$.

So $S = [-4; 1]$.

1.3 Radicals

1.3.1 Square root

Definition 1.3.1 The square root of a positive real numbers a is the positive real numbers b such that:

$$b^2 = a; \quad a \geq 0 \Leftrightarrow b = \sqrt{a} = a^{\frac{1}{2}}$$

Proposition 1.3.1 .

$$1. \sqrt{ab} = \sqrt{a}\sqrt{b}.$$

$$2. \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}, b \neq 0$$

$$3. (\sqrt{a})^2 = a$$

$$4. \sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$

Example 1.3.1 Solve in $\mathbb{R}$: $x^2 = 25$.

we have $x^2 = 25 \Leftrightarrow x = \pm\sqrt{25} \Leftrightarrow x = 5$ or $x = -5$.

Example 1.3.2 Solve in $\mathbb{R}$: $\sqrt{2-x} = x$.

Firstly, $\sqrt{2-x}$ must be well defined $\Leftrightarrow 2-x \geq 0 \Leftrightarrow x \leq 2 \Leftrightarrow x \in]-\infty; 2]$

Furthermore, x must be positive (because $\sqrt{2-x} = x$)

$$\Rightarrow x \in [0; 2].$$

Now, $\sqrt{2-x} = x \Leftrightarrow 2-x = x^2 \Leftrightarrow x^2 + x - 2 = 0 \Leftrightarrow x_1 = 1$ or $x_2 = -2$.

Since x_1 is the only element in the interval $[0; 2]$, our equation has only one solution, which is $x = 1$.

1.3.2 Cube root

Definition 1.3.2 The cube root of a real numbers a is the real numbers b such that:

$$b^3 = a \Leftrightarrow b = \sqrt[3]{a} = a^{\frac{1}{3}}$$

Example 1.3.3 Solve in $\mathbb{R}$: $x^3 = -27 \Leftrightarrow x = \sqrt[3]{-27} = -3$. So $S = -3$.

Example 1.3.4 Solve in $\mathbb{R}$: $\sqrt[3]{2x-4} = 2 \Leftrightarrow 2x-4 = 8 \Leftrightarrow x = 6$. So $S = 6$.

1.3.3 n-th root

Definition 1.3.3 The n -th root of a real numbers a is the real numbers b such that:

$$b^n = a \Leftrightarrow b = \sqrt[n]{a} = a^{\frac{1}{n}}, n \in \mathbb{N}.$$

Remark 1.3.1 .

- If n is even $\Rightarrow b = \pm\sqrt[n]{a}; a \geq 0$
- If n is odd $\Rightarrow b = \sqrt[n]{a}; a \in \mathbb{R}$

Proposition 1.3.2 .

1. $\sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$.
2. $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}, b \neq 0$
3. $(\sqrt[n]{a})^m = a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
4. $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$.
5. If n is odd, $(\sqrt[n]{a})^n = \sqrt[n]{a^n} = a$.
6. If n is even, $(\sqrt[n]{a})^n = a$.
7. If n is even, $\sqrt[n]{a^n} = |a|$.

Example 1.3.5 . Simplify

$$\begin{aligned}\sqrt[3]{64m^3} &= \sqrt[3]{64} \sqrt[3]{m^3} = \sqrt[3]{4^3} \sqrt[3]{m^3} = 4m. \\ \sqrt[3]{100x^{16}} &= \sqrt[3]{10^2} \sqrt[3]{x^{16}} = \sqrt[3]{10^2} \sqrt[3]{(x^8)^2} = 10x^8. \\ \sqrt{25a^2b^6} &= 5\sqrt{a^2} \sqrt{(b^3)^2} = 5|a||b^3| = |5ab^3|.\end{aligned}$$

Example 1.3.6 Solve in $\mathbb{R}$ the equations

$$\begin{aligned}1) \quad 12x^5 &= 384 \Leftrightarrow x^5 = 32 \Leftrightarrow x = \sqrt[5]{32} \Leftrightarrow x = \sqrt[5]{2^5} = 2. \\ 2) \quad \left(\frac{x+9}{5}\right)^2 &= \sqrt{144 \cdot 3^2} \Leftrightarrow \left(\frac{x+9}{5}\right)^2 = 36 \Leftrightarrow \frac{x+9}{5} = 6 \quad \text{or} \quad \frac{x+9}{5} = -6 \\ &\Leftrightarrow x = 21 \quad \text{or} \quad x = -39. \\ \text{So } S &= \{-39, 21\}\end{aligned}$$

1.3.4 Integer part (Floor function)

Definition 1.3.4 Let $x \in \mathbb{R}$, the integer part of x is the unique integer, denoted $E(x)$ satisfying:

$$E(x) \leq x < E(x) + 1$$

Example 1.3.7 $E(2,452) = 2$; $E(\pi) = E(3,14...) = 3$; $E(-3,4) = -4$; $E(x) = 3 \Leftrightarrow 3 \leq x < 4$.

Example 1.3.8 Solve in $\mathbb{R}$ the equations

$$\begin{aligned}\bullet \quad E\left(\frac{x-1}{2}\right) &= -2 \Leftrightarrow E\left(\frac{x-1}{2}\right) \leq \frac{x-1}{2} < E\left(\frac{x-1}{2}\right) + 1 \Leftrightarrow -2 \leq \frac{x-1}{2} < -2 + 1 \\ &\Leftrightarrow -4 \leq x-1 < -2 \Leftrightarrow -3 \leq x < -1. \\ \text{So } S &= [-3; -1[.\end{aligned}$$

- $E(2x) = x - 1 \Leftrightarrow x - 1 \leq 2x < x - 1 + 1 \quad \text{and} \quad x - 1 \in \mathbb{Z}$
 $\Leftrightarrow x - 1 \leq 2x \quad \text{and} \quad 2x < x \quad \text{and} \quad x \in \mathbb{Z} \Leftrightarrow x \geq -1 \quad \text{and} \quad x < 0 \quad \text{and} \quad x \in \mathbb{Z}$
 $\Leftrightarrow x \in [-1; 0[\quad \text{and} \quad x \in \mathbb{Z} \Leftrightarrow x = -1.$
 So $S = \{-1\}.$

1.4 Vectors

1.4.1 Definition of a vector space

Definition 1.4.1 A vector space E is a set of vectors equipped with two operations

- vector addition (an internal operation denoted by $(+)$)

$$\begin{aligned} (+) : E \times E &\rightarrow E \\ (u, v) &\rightarrow u + v \end{aligned}$$

- vector multiplication (an external operation noted $(\cdot)$),

$$\begin{aligned} (\cdot) : \mathbb{K} \times E &\rightarrow E \\ (\lambda, V) &\rightarrow \lambda V \end{aligned}$$

$\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$

These operation must satisfy the following:

- $\forall u, v \in E : u + v = v + u$
- $\forall u, v, w \in E : u + (v + w) = (u + v) + w.$
- $\exists e = 0 \in E, \forall u \in E : u + e = e + u = u.$
- $\forall u \in E, \exists (-u) \in E : u + (-u) = (-u) + u = 0.$
- $\forall \lambda \in \mathbb{K}, \forall u, v \in E : \lambda \cdot (u + v) = \lambda \cdot u + \lambda \cdot v.$
- $\forall \lambda, \mu \in \mathbb{K}, \forall u \in E : (\lambda + \mu) \cdot u = \lambda \cdot u + \mu \cdot u.$
- $\forall \lambda, \mu \in \mathbb{K}, \forall u \in E : (\lambda \cdot \mu) \cdot u = \lambda \cdot (\mu \cdot u).$
- $\forall u \in E, 1_{\mathbb{K}} \cdot u = u.$

1.4.2 Example of vector spaces

1. $(\mathbb{R}, +, \cdot)$ is an $\mathbb{R}$ -vector space.

2. $(\mathbb{R}^2, +, \cdot)$ is an $\mathbb{R}$ -vector space.

The plane $\mathbb{R}^2$: a vector is (x, y) , and we define

$$\begin{cases} (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2), \\ \lambda \cdot (x, y) = (\lambda x, \lambda y). \end{cases}$$

3. $(\mathbb{R}^3, +, \cdot)$ is an $\mathbb{R}$ -vector space. The space $\mathbb{R}^3$: a vector is (x, y, z) , and we define

$$\begin{cases} (x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2), \\ \lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda z). \end{cases}$$

4. $(\mathbb{R}^n, +, \cdot)$ is an $\mathbb{R}$ -vector space.

$$(+): \mathbb{R}^n \cdot \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n).$$

$$(.): \mathbb{R}^n \cdot \mathbb{R}^n \rightarrow \mathbb{R}^n$$

$$\lambda \cdot (y_1, y_2, \dots, y_n) = (\lambda y_1, \lambda y_2, \dots, \lambda y_n).$$

1.4.3 Linear dependence and independence

Definition 1.4.2 (Linear Independence) Let $(E, +, \cdot)$ be a vector space and let $x_1, x_2, \dots, x_n \in E$ be vectors. We say that the vectors $x_1, x_2, \dots, x_n$ are linearly independent if for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$ the equation

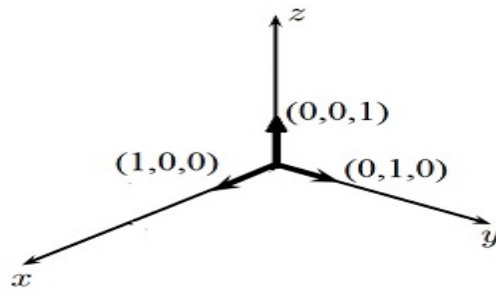
$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0_E$$

can only be satisfied if $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0_{\mathbb{R}}$.

Remark 1.4.1 Any set consists of a single vector $x \neq 0_E$ is linearly independent.

Example 1.4.1 1. In the vector space $(\mathbb{R}^3, +, \cdot)$ over $\mathbb{R}$, we consider the vectors

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1).$$



Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If

$$\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0_{\mathbb{R}^3},$$

then

$$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = (0,0,0) \iff (\alpha_1, \alpha_2, \alpha_3) = (0,0,0).$$

So,

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, the vectors e_1, e_2 , and e_3 are linearly independent.

2. In the vector space $(\mathbb{R}^3, +, \cdot)$ over $\mathbb{R}$, we consider the vectors

$$v_1 = (0,1,-1), \quad v_2 = (1,0,-1), \quad v_3 = (1,1,0).$$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^3},$$

then

$$\alpha_1(0,1,-1) + \alpha_2(1,0,-1) + \alpha_3(1,1,0) = (0,0,0),$$

which gives

$$(\alpha_2 + \alpha_3, \alpha_1 + \alpha_3, -\alpha_1 - \alpha_2) = (0,0,0).$$

Thus, we obtain the system

$$\begin{cases} \alpha_2 + \alpha_3 = 0, \\ \alpha_1 + \alpha_3 = 0, \\ -\alpha_1 - \alpha_2 = 0, \end{cases}$$

which implies

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

Hence, the vectors v_1, v_2 , and v_3 are linearly independent.

Definition 1.4.3 (Linear Dependence) Let $(E, +, \cdot)$ be a vector space and let $x_1, x_2, \dots, x_n \in E$ be vectors. We say that the vectors $x_1, x_2, \dots, x_n$ are linearly dependent if there exist $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{R}$, not all equal to zero (i.e. at least one $\alpha_i \neq 0$, for $i = 1, \dots, n$), such that

$$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0_E.$$

Example 1.4.2 In the vector space $(\mathbb{R}^2, +, \cdot)$ over $\mathbb{R}$ we consider the vectors

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad v_3 = (2, 3).$$

Let $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$. If

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0_{\mathbb{R}^2},$$

then

$$\alpha_1(1, 0) + \alpha_2(0, 1) + \alpha_3(2, 3) = (0, 0),$$

which leads to

$$(\alpha_1 + 2\alpha_3, \alpha_2 + 3\alpha_3) = (0, 0).$$

Thus, we obtain the system

$$\begin{cases} \alpha_1 + 2\alpha_3 = 0, \\ \alpha_2 + 3\alpha_3 = 0, \end{cases} \quad \implies \quad \alpha_1 = -2\alpha_3, \quad \alpha_2 = -3\alpha_3.$$

For example, if $\alpha_3 = 1$, then $\alpha_1 = -2$ and $\alpha_2 = -3$. Hence

$$(-2)(1, 0) + (-3)(0, 1) + (1)(2, 3) = (0, 0).$$

Therefore, the vectors v_1, v_2 , and v_3 are linearly dependent.

Remark 1.4.2 1. If one of the vectors $x_1, x_2, \dots, x_n$ equals the zero vector 0_E , then $x_1, x_2, \dots, x_n$ are linearly dependent.

2. If the number of vectors is greater than the dimension of the space or plane, then the vectors are linearly dependent.

Example 1.4.3 In the vector space $(\mathbb{R}^3, +, \cdot)$ over $\mathbb{R}$ we consider the vectors

$$v_1 = (2, -2, -1), \quad v_2 = (1, 1, 1), \quad v_3 = (1, 2, 3), \quad v_4 = (2, -1, 1).$$

Four vectors in a space of dimension three are linearly dependent. Because we have

$$(2, -2, -1) = 1 \cdot (1, 1, 1) - 1 \cdot (1, 2, 3) + 1 \cdot (2, -1, 1).$$

So, v_1 is a linear combination of v_2, v_3 and v_4 . Hence, the set

$$\{v_1, v_2, v_3, v_4\}$$

is linearly dependent.

1.5 Bases and Dimension

1.5.1 Bases

Definition 1.5.1 Let $(E, +, -)$ be a k -vector space and let $v_1, v_2, \dots, v_n$ be vectors in E . We say that the set $\{v_1, v_2, \dots, v_n\}$ form a basis of E if

1. $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

2. $\{v_1, v_2, \dots, v_n\}$ generates E i.e

$$\forall v \in E, \quad \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{K} \quad v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$

Example 1.5.1 In the vector space $(\mathbb{R}^2, +, \cdot)$ over $\mathbb{R}$ the set

$$B = \{v_1 = (1; 0), \quad v_2 = (1; -1)\},$$

form a basis of $\mathbb{R}^2$. Indeed

1. Let $\alpha_1, \alpha_2 \in \mathbb{R}$. If

$$\alpha_1 v_1 + \alpha_2 v_2 = 0_{\mathbb{R}^2},$$

then

$$\alpha_1(1;0) + \alpha_2(1;-1) = (0,0),$$

which leads to

$$(\alpha_1 + 2\alpha_2, -\alpha_2) = (0,0).$$

Thus, we obtain the system

$$\begin{cases} \alpha_1 + \alpha_2 = 0, \\ -\alpha_2 = 0, \end{cases} \quad \Rightarrow \quad \alpha_1 = \alpha_2 = 0.$$

So, $\{v_1; v_2\}$ is linearly independent.

2. $\{v_1; v_2\}$ generates $\mathbb{R}^2$ i.e

$$\forall v = (x; y) \in \mathbb{R}^2, \quad \exists \lambda_1, \lambda_2 \in \mathbb{R} \quad v = (x; y) = \lambda_1 v_1 + \lambda_2 v_2.$$