

Series N°1 – Part I Real Numbers Set

Exercise 1. (Use course supplement and Appendix 1 of chapter 1-part I)

1. Show that $\forall x, y \in \mathbb{R}, x^2 + y^2 = 0 \iff x = y = 0$
2. Show that $\forall x, y \in \mathbb{R}, x \neq 2 \wedge y \neq 2 \implies 2x - xy + 2y - 2 \neq 2$.
3. Let x and y be two real distinct numbers of 1. Show that if $x \neq y$, then $\frac{1}{x-1} \neq \frac{1}{y-1}$.
4. Show that n is prime $\implies n = 2 \vee n$ is odd.

Exercise 2.

1. Show by contradiction that $\forall x \in \mathbb{R}^*, \frac{9+x}{2x} \neq \frac{1}{2}$.
2. Show by induction that $\forall n \geq 0, 6^n + 9$ is a multiple of 5.

Exercise 3. In each case, give the smallest set to which the following numbers:

$$\frac{125}{5}, \quad \frac{7}{5}, \quad \frac{21}{12}, \quad \frac{-35}{7}, \quad \frac{14}{21}, \quad \pi, \quad \sqrt{2}$$

Exercise 4. 1. Which set does the number $\left(\sqrt{n} + \sqrt{\frac{1}{n}}\right)^2$ belong to, where $n \in \mathbb{N}^*$?

2. Show that the square of an odd integer is odd.
3. Verify that when n is a natural number, the number $\frac{n(n^2+1)}{2}$ is also a natural number.

Exercise 5. Solve the following equations and inequalities in $\mathbb{R}$:

- 1) $(x-2)(3x-1) > 3x-1,$
- 2) $\frac{x^2-x+2}{x+1} \leq 1,$
- 3) $|5x+2| = -\frac{7}{2},$
- 4) $2|x+2| + |x-5| = 9,$
- 5) $|x+12| = |x^2-8|,$
- 6) $|2x-11| < |x-5|,$
- 7) $|x+3| \leq x,$
- 8) $\left|\frac{2x-1}{x+1}\right| = 1, .$

Exercise 6. 1. Show that $\forall a, b \in \mathbb{R}^+; (\sqrt{a} + \sqrt{b})^2 = a + b + 2\sqrt{ab}$

2. Solve the equation $\sqrt{41-x} + \sqrt{41+x} = 10$

Exercise 7. Solve in $\mathbb{R}$

- 1) $(x-1)^3 + 27 = 0,$
- 2) $\sqrt[5]{3x-4} = 2$
- 3) $\sqrt{2-x} = x$
- 4) $\sqrt{2x+3} \geq x,$
- 5) $\sqrt{x^2-16} \geq -2.$

Exercise 8. The floor of a real number x is denoted by $[x]$ or $E(x)$.

1. Evaluate $[2], [-7], [3.7], [-3.7], [\pi], \left[-\frac{3}{8}\right], [\sqrt{2}].$
2. Solve in $\mathbb{R}$

- 1) $E\left(\frac{x-1}{2}\right) = -2,$
- 2) $E(2x) = x-1,$
- 3) $2([x])^2 - 3[x] + 1 = 0.,$
- 4) $E\left(\frac{1}{x}\right) = 3,$
- 5) $E(x) + |x-1| = x.$

Course Supplement

1 Some Types of Mathematical Reasoning

1.1 Direct Reasoning

To prove $P \Rightarrow Q$, assume that P is true and show that Q is true.

1.2 Contraposition

We use the equivalence $(P \Rightarrow Q) \iff (\overline{Q} \Rightarrow \overline{P})$. Thus, to prove $P \Rightarrow Q$, it is enough to prove $\overline{Q} \Rightarrow \overline{P}$.

1.3 Contradiction

To prove P , assume that $\overline{P}$ is true and derive a contradiction. Therefore, $\overline{P}$ is false and P is true.

1.4 Induction

Induction is used to prove that a proposition $P(n)$ is true for all $n \in \mathbb{N}$.

- **Initialization:** Prove that $P(0)$ is true.
- **Heredity:** Assume $P(n)$ is true and prove $P(n+1)$.
- **Conclusion:** Therefore, $P(n)$ is true for all $n \in \mathbb{N}$.

Remark. If the proposition is required for all $n \geq n_0$, start the initialization at n_0 .

Additional exercises

Exercise 1

1. Let $a, b \geq 0$. Show that if $\frac{a}{1+b} = \frac{b}{1+a}$ then $a = b$.
2. Let $n \in \mathbb{N}$. Show that if n^2 is even then n is even.
3. Show by induction that $\forall n \in \mathbb{N}, \quad 2^n > n..$
4. Let's show the following proposition: 0 has no inverse in $\mathbb{R}$

Exercise 2

1. Let a and b two real numbers in $\mathbb{R}_+^*$, with $a \leq b$.
Compare between a^2 and b^2 , then between $\frac{1}{a}$ and $\frac{1}{b}$.
2. Show that if a and b are strictly positive, then $\frac{a}{b} + \frac{b}{a} \geq 2$. Deduce that
$$(a+b) \left(\frac{1}{a} + \frac{1}{b} \right) \geq 4.$$
3. Prove the following relation $\forall x, y \in \mathbb{R}$,
$$x^2 + y^2 \geq 2xy,$$
then deduce that $\forall x, y \in \mathbb{R}$,
$$(x+y)^2 \geq 4xy.$$