Series 2: Real-Valued Functions

Exercise 1 1) Solve in $\mathbb{R}$ the following equations:

$$1)\ \sin x = \frac{1}{2}, \qquad 2)\ \cos x = \frac{\sqrt{3}}{2}, \qquad 3)\ \tan x = \sqrt{3},$$

$$4)\ \sin x \cos x = \frac{1}{4}, \qquad 5)\ \sin\left(2x - \frac{\pi}{3}\right) = \cos\left(\frac{x}{3}\right).$$

2) Solve in $[0, 2\pi]$ the following equations: $a)\quad 2\sin x - 1 = 0, \qquad b)\quad \cos x = \dfrac{\sqrt{2}}{2}.$

Exercise 2 Solve the following inequalities on the interval $[0, 2\pi]$

$$1)\quad \sin(x) \geq \frac{1}{2}, \qquad 2)\quad \cos(x) > \frac{\sqrt{2}}{2}.$$

Exercise 3 We define on $\mathbb{R}$ the **hyperbolic cosine function** denoted by cosh, as follows:

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

We define on $\mathbb{R}$ the **hyperbolic sine function** denoted by sinh, as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

1. Show that the function cosh is even and the function sinh is odd.
2. Show that for all $x \in \mathbb{R}$, $\cosh^2(x) - \sinh^2(x) = 1$.
3. Show that the derivative of sinh is cosh and the derivative of cosh is sinh.
4. Show that $\text{Range}(\sinh x) = \mathbb{R}$, $\text{Range}(\cosh x) = [1, +\infty[$, $\text{Range}(\tanh x) =]-1, 1[$.

Remark: We define also as in trigonometry the **hyperbolic tangent** tanh and the **hyperbolic cotangent** coth as follow: $\tanh(x) = \dfrac{\sinh(x)}{\cosh(x)}$ and $\coth(x) = \dfrac{\cosh(x)}{\sinh(x)}$.

Exercise 4 Solve the following equations:

$$\sinh(2x) = \frac{3}{4}. \qquad 3\cosh(x) - \sinh(x) - 3 = 0, \qquad \tanh(x) = \frac{3}{5}.$$

Exercise 5 Calculate the following limits :

$$1)\ \lim_{x\to 0} \frac{\ln(1+ax)}{x}, \quad 2)\ \lim_{x\to 0} \frac{e^{2x}-1}{x}, \quad 3)\ \lim_{x\to 0} \frac{\sin(ax)}{x}, \quad 4)\ \lim_{x\to 0} \ln(1+x^2)\sin\frac{1}{x^2},$$

$$5)\ \lim_{x\to 0} \frac{\ln(1+x^2)}{\sin^2 x}, \quad 6)\ \lim_{x\to 0} \frac{1-\cos(2x)}{x^2}, \quad 7)\ \lim_{x\to+\infty} \frac{\cosh x}{\sinh x}, \quad 8)\ \lim_{x\to 1} \frac{\sqrt{x}-1}{1-x}.$$

Exercise 6 Let us define the function f by

$$f(x) = \begin{cases} a(x+1) - b\ln(e-x), & \text{if } x < 0, \\ -3, & \text{if } x = 0, \\ 2a + b\cos x, & \text{if } x > 0. \end{cases}$$

Determine the values of a and b for which the function f is continuous on $\mathbb{R}$.

Exercise 7

1) Show that the equation $x^3+6x+12 = 0$ has a unique solution in $\mathbb{R}$. Then, verify that $\alpha \in]-1,48; -1,47[$
2) Show that the equation $x^7 - x^2 + 1 = 0$, has at least one solution in the interval $I = [-2, 0]$.

Exercise 8 Let the function defined by:

$$f(x) = \arccos(2x-1) - \arcsin(3x^2)$$

1. Determine the domain of definition of $f(x)$.
2. Calculate the derivative of $f(x)$.

Exercise 9 Show that for all $x > 0$:

$$\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}.$$

Exercise 10 Solve the following equations:

1. $\arcsin x = \arcsin \frac{2}{5} + \arcsin \frac{3}{5}$. (Using the formula $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a)$)
2. $\arctan 2x + \arctan 3x = \frac{\pi}{4}$. (Using the formula $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$).

Exercise 11 1) Solve the equation, for $x \in]-1,1[$, $\operatorname{artanh}(x) = \ln(3)$.
2) Determine the domain of definition of the function f defined by

$$f(x) = \operatorname{arccosh}\left(\frac{x^2+1}{2x}\right),$$

simplify its expression.

Additional Exercises

Exercise 1 1) Show that $\text{Range}(\tanh x) =]-1,1[$, and $\text{Range}(\coth x) =]-\infty,-1[\cup]1,+\infty[$.
2) Simplify the following expressions: a) $\frac{2\cosh^2 x - \sinh 2x}{x + ln(\cosh x) + ln2}$, b) $\tanh(\frac{1}{2}ln3)$.
3) Solve in $\mathbb{R}$ the equations:

a) $2\cos^2(x) + 9\cos(x) + 4 = 0$. b) $\cosh(x) = \frac{13}{5}$, c) $\tanh(x) = \frac{3}{5}$, d) $\arccos x = 2\arccos\frac{3}{4}$.

4) Solve the inequalities on the interval $[0, 2\pi]$: a) $\sin(x) > -\frac{1}{2}$, b) $\cos(x) \geq \frac{1}{2}$.

Exercise 2 For each of the following cases, determine α so that the function is continuous on $\mathbb{R}$:

1) $f(x) = \begin{cases} \frac{x^2-x}{x}, & \text{if } x \neq 0, \\ \alpha, & \text{if } x = 0, \end{cases}$ 2) $f(x) = \begin{cases} \frac{\sqrt{x^2-x+1}-x}{x-1}, & \text{if } x \neq 1, \\ \alpha, & \text{if } x = 1. \end{cases}$

Exercise 3 I) Let f be the function defined on $\mathbb{R}$ by: $f(x) = 2x^3 - 3x^2 - 1$

1. Study the variations of the function f.
2. Show that the equation $f(x) = 4$ has a unique solution α on $]2;+\infty[$.

II) Show that the equation $e^x - \ln(1+x) - 2 = 0$ has at least one solution in the interval $I = [0, 1]$.
Exercise 4 1) Prove that, for all x in $[-1,1]$, $\sin(\arccos(x)) = \sqrt{1-x^2}$ and $\cos(\arcsin(x)) = \sqrt{1-x^2}$.
2) Show that for all $x \in]-1;1[$: $\arcsin x + \arccos x = \frac{\pi}{2}$.
3) Simplify the following expressions: $\cos(2\arcsin(x))$, $\sin(2\arcsin(x))$, $\cosh(\operatorname{arcsinh}(x))$ deduce that: $\tanh(\operatorname{arcsinh}(x))$.
Exercise 5 By using the definitions of hyperbolic functions in terms of exponentials, prove that : for $x \in]-1,1[$, $\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$.