

CONTENTS



1	Numbers and Vectors	2
1.1	Bases and Dimension	2
1.1.1	Bases	2
1.1.2	Definition of Dimension	4
1.2	Vector Projection	5
1.3	Plane in Polar Coordinates (2D)	6
1.4	Cylindrical Coordinates (3D)	7
2	Real-valued function	9
2.1	Review of elementary function	9
2.1.1	Exponential function	9
2.1.2	Logarithm function	10
2.1.3	Exponential of base a	11
2.2	Trigonometric functions	11
2.2.1	Sine and Cosine function	11
2.2.2	Solution of trigonometric Equations	12
2.3	Hyperbolic Function	13
2.3.1	Hyperbolic sine and cosine function	13
2.3.2	Hyperbolic tangent and cotangent function	14

NUMBERS AND VECTORS



1.1 Bases and Dimension

1.1.1 Bases

Definition 1.1.1 Let $(E, +, -)$ be a k -vector space and let $v_1, v_2, \dots, v_n$ be vectors in E . We say that the set $\{v_1, v_2, \dots, v_n\}$ form a basis of E if

1. $\{v_1, v_2, \dots, v_n\}$ is linearly independent.

2. $\{v_1, v_2, \dots, v_n\}$ generates E i.e

$$\forall v \in E, \quad \exists \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{K} \quad v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n.$$

Example 1.1.1 “*latex* In the vector space $(\mathbb{R}^2, +, \cdot)$ over $\mathbb{R}$, the set

$$B = \{v_1 = (1, 0), \quad v_2 = (1, -1)\}$$

forms a basis of $\mathbb{R}^2$. Indeed,

1. Linear independence

Let $\alpha_1, \alpha_2 \in \mathbb{R}$. If

$$\alpha_1 v_1 + \alpha_2 v_2 = 0_{\mathbb{R}^2},$$

then

$$\alpha_1(1, 0) + \alpha_2(1, -1) = (0, 0),$$

which gives

$$(\alpha_1 + \alpha_2, -\alpha_2) = (0, 0).$$

Thus, we obtain the system

$$\begin{cases} \alpha_1 + \alpha_2 = 0, \\ -\alpha_2 = 0. \end{cases}$$

Hence,

$$\alpha_2 = 0 \quad \text{and} \quad \alpha_1 = 0.$$

Therefore, the family $\{v_1, v_2\}$ is linearly independent.

2. **Spanning** $\mathbb{R}^2$

Let $v = (x, y) \in \mathbb{R}^2$. We seek $\lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$v = \lambda_1 v_1 + \lambda_2 v_2.$$

This is equivalent to

$$(x, y) = \lambda_1(1, 0) + \lambda_2(1, -1),$$

which gives the system

$$\begin{cases} \lambda_1 + \lambda_2 = x, \\ -\lambda_2 = y. \end{cases}$$

From the second equation, we obtain

$$\lambda_2 = -y.$$

Substituting into the first equation yields

$$\lambda_1 - y = x,$$

so

$$\lambda_1 = x + y.$$

Therefore, for every $(x, y) \in \mathbb{R}^2$,

$$(x, y) = (x + y)v_1 - yv_2.$$

Hence, $\{v_1, v_2\}$ generates $\mathbb{R}^2$.

Since the family B is linearly independent and generates $\mathbb{R}^2$, we conclude that

$$B = \{(1, 0), (1, -1)\} \text{ is a basis of } \mathbb{R}^2.$$

Remarks

- 1) If one of the vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is the zero vector, then the family $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is linearly dependent.
- 2) If the number of vectors is greater than the dimension of the vector space, then the vectors are linearly dependent.

Example

Consider the vectors in $\mathbb{R}^2$:

$$\vec{u} = (1, 0), \quad \vec{v} = (0, 1), \quad \vec{w} = (2, 3).$$

The family $\{\vec{u}, \vec{v}, \vec{w}\}$ is linearly dependent because it contains three vectors, while

$$\dim(\mathbb{R}^2) = 2.$$

1.1.2 Definition of Dimension

Let $(E, +, \cdot)$ be a vector space and let $\mathcal{B}$ be a basis of E .

The dimension of E , denoted by $\dim(E)$, is the number of vectors in the basis $\mathcal{B}$.

Examples

and

$$\dim(\mathbb{R}^2) = 2$$

$$\dim(\mathbb{R}^3) = 3.$$

1.2 Vector Projection

Let

$$\vec{a} = (x_a, y_a, z_a), \quad \vec{b} = (x_b, y_b, z_b), \quad \vec{b} \neq \vec{0}.$$

The projection of $\vec{a}$ onto $\vec{b}$ is given by

$$\text{Proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}$$

where

$$\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b + z_a z_b$$

and

$$\|\vec{b}\| = \sqrt{x_b^2 + y_b^2 + z_b^2}.$$

Example 1

Let

$$\vec{a} = (3, 4), \quad \vec{b} = (2, 0).$$

The projection vector is

$$\text{Proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}.$$

First, we calculate the dot product:

$$\vec{a} \cdot \vec{b} = 3 \times 2 + 4 \times 0 = 6.$$

Next,

$$\|\vec{b}\|^2 = 2^2 + 0^2 = 4.$$

Therefore,

$$\begin{aligned} \text{Proj}_{\vec{b}}(\vec{a}) &= \frac{6}{4}(2, 0) \\ &= \frac{3}{2}(2, 0) \\ &= (3, 0). \end{aligned}$$

Hence,

$$\vec{p} = (3, 0).$$

Example 2

Let

$$\vec{a} = (2, 3, 1), \quad \vec{b} = (1, 0, 2).$$

The projection vector is

$$\text{Proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b}.$$

First, we calculate the dot product:

$$\vec{a} \cdot \vec{b} = 2 \times 1 + 3 \times 0 + 1 \times 2 = 4.$$

Next,

$$\|\vec{b}\|^2 = 1^2 + 0^2 + 2^2 = 5.$$

Therefore,

$$\begin{aligned} \text{Proj}_{\vec{b}}(\vec{a}) &= \frac{4}{5}(1, 0, 2) \\ &= \left(\frac{4}{5}, 0, \frac{8}{5}\right). \end{aligned}$$

Hence,

$$\vec{p} = \left(\frac{4}{5}, 0, \frac{8}{5}\right).$$

“*latex*

1.3 Plane in Polar Coordinates (2D)

The point (x, y) is expressed in Cartesian coordinates, while (r, θ) represents its polar coordinates.

The relations between the two systems are:

$$\begin{aligned} x &= r \cos \theta, & y &= r \sin \theta, \\ r &= \sqrt{x^2 + y^2}, & \theta &= \arctan\left(\frac{y}{x}\right). \end{aligned}$$

Example:

Let the point have the following Cartesian coordinates:

$$(x, y) = (3, 3).$$

Determine its polar coordinates (r, θ) .

Method 1:

First, calculate the radius:

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{3^2 + 3^2} \\ &= \sqrt{18} \\ &= 3\sqrt{2}. \end{aligned}$$

Next, calculate the angle:

$$\begin{aligned} \theta &= \arctan\left(\frac{y}{x}\right) \\ &= \arctan\left(\frac{3}{3}\right) \\ &= \frac{\pi}{4}. \end{aligned}$$

Therefore,

$$(r, \theta) = \left(3\sqrt{2}, \frac{\pi}{4}\right).$$

Method 2:

We use the following relations:

$$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}.$$

Thus,

$$\begin{aligned} \cos \theta &= \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}, \\ \sin \theta &= \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}. \end{aligned}$$

Consequently,

$$\theta = \frac{\pi}{4}.$$

1.4 Cylindrical Coordinates (3D)

The point (x, y, z) is expressed in Cartesian coordinates, while (r, θ, z) represents its cylindrical coordinates.

The relations are:

$$\begin{aligned}x &= r \cos \theta, \\y &= r \sin \theta, \\z &= z, \\r &= \sqrt{x^2 + y^2}, \\\theta &= \arctan\left(\frac{y}{x}\right).\end{aligned}$$

Example:

Let the point have the following Cartesian coordinates:

$$(x, y, z) = (2, 2, 5).$$

Determine its cylindrical coordinates (r, θ, z) .

Calculation of the radius:

$$\begin{aligned}r &= \sqrt{x^2 + y^2} \\&= \sqrt{2^2 + 2^2} \\&= \sqrt{8} \\&= 2\sqrt{2}.\end{aligned}$$

Calculation of the angle:

$$\begin{aligned}\theta &= \arctan\left(\frac{y}{x}\right) \\&= \arctan\left(\frac{2}{2}\right) \\&= \frac{\pi}{4}.\end{aligned}$$

Finally,

$$z = 5.$$

Therefore, the cylindrical coordinates are:

$$(r, \theta, z) = \left(2\sqrt{2}, \frac{\pi}{4}, 5\right).$$



2.1 Review of elementary function

2.1.1 Exponential function

Definition 2.1.1 We call exponential function the unique differentiable function on $\mathbb{R}$ such that $f' = f$ and $f(0) = 1$, we denote this function by

$$\exp : \mathbb{R} \rightarrow]0; +\infty[\quad (2.1)$$

$$x \mapsto \exp(x) \quad (2.2)$$

Remark 2.1.1 For $x \in \mathbb{R}$, we also write e^x for $\exp(x)$

Proposition 2.1.1 $\forall x, y \in \mathbb{R}$

$$1. e^{x+y} = e^x e^y$$

$$2. e^{x-y} = \frac{e^x}{e^y}$$

$$3. (e^x)^n = e^{nx}$$

$$4. e^{-x} = \frac{1}{e^x}$$

$$5. \exp : \mathbb{R} \rightarrow]0; +\infty[, \text{ is a continuous and differentiable on } \mathbb{R} \text{ and } (e^x)' = e^x.$$

$$6. \text{ the exponential function is strictly increasing on } \mathbb{R}.$$

$$7. \forall x \in \mathbb{R}, e^x > 0$$

$$\begin{array}{ll}
 \lim_{x \rightarrow +\infty} e^x = +\infty & \lim_{x \rightarrow -\infty} e^x = 0 \\
 \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 & \lim_{x \rightarrow 0} \frac{e^{ax} - 1}{x} = a \\
 \lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x = e & \lim_{x \rightarrow 0} (1 + x)^{\frac{1}{x}} = e \\
 \lim_{x \rightarrow +\infty} \frac{e^x}{x^n} = +\infty & \lim_{x \rightarrow -\infty} x^n e^x = 0
 \end{array}$$

2.1.2 Logarithm function

Definition 2.1.2 The inverse function of $\exp : \mathbb{R} \rightarrow]0; +\infty[$ is called a logarithm function $\ln$ and defined by

$$\ln :]0; +\infty[\rightarrow \mathbb{R} \quad (2.3)$$

$$x \rightarrow \ln x \quad (2.4)$$

Proposition 2.1.2 $\forall x, y \in \mathbb{R}_+^*$

1. $\ln(xy) = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln \frac{1}{x} = -\ln x$
4. $\ln x^n = n \ln x$
5. $\ln :]0; +\infty[\rightarrow \mathbb{R}$, is a continuous and differentiable on $]0; +\infty[$ and $(\ln(x))' = \frac{1}{x}$.
6. the function $\ln$ is strictly increasing on $]0; +\infty[$.

Remark 2.1.2 1. $\forall x \in \mathbb{R}, \ln e^x = x$.

2. $\forall x \in]0; +\infty[, e^{\ln x} = x$.

2.1.3 Exponential of base a

Definition 2.1.3 For any positive number $a > 0$, and $a \neq 1$, an exponential function of base a is defined by

$$f : \mathbb{R} \rightarrow]0; +\infty[\quad (2.5)$$

$$x \rightarrow a^x = e^{x \ln a} \quad (2.6)$$

The function $x \rightarrow a^x$ defined on $\mathbb{R}$ to $]0; +\infty[$, continuous, differentiable and satisfies:

1. $(a^x)' = (e^{x \ln a})' = \ln a e^{x \ln a} = a^x \ln a, \forall x \in \mathbb{R}$.
2. For $a > 1$, the function $x \rightarrow a^x$ is strictly increasing and $\lim_{x \rightarrow +\infty} a^x = +\infty$.
3. For $0 < a < 1$, the function $x \rightarrow a^x$ is strictly decreasing and $\lim_{x \rightarrow +\infty} a^x = 0$.

2.2 Trigonometric functions

2.2.1 Sine and Cosine function

Function	$\sin(x)$	$\cos(x)$
definition of domain	$\mathbb{R}$	$\mathbb{R}$
Parity	odd	even
Period	$T = 2\pi$	$T = 2\pi$
Derivative	$\cos(x)$	$-\sin(x)$
Range	$[-1; 1]$	$[-1; 1]$

Particular limits

$$\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 1;$$

$$\lim_{x \rightarrow 0} \frac{1 - \cos(x)}{x^2} = \frac{1}{2}.$$

Properties: The sine and cosine functions satisfy the following properties, $\forall x \in \mathbb{R}$

1. $\cos^2 x + \sin^2 x = 1$
2. $\cos^2 x = \frac{1 + \cos 2x}{2}$
3. $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$4. \sin(2x) = 2 \sin x \cos x$$

$$5. \cos(2x) = \cos^2 x - \sin^2 x = -1 + 2 \cos^2 x = 1 - 2 \sin^2 x$$

Addition formula

- $\sin(a + b) = \sin a \cdot \cos b + \sin b \cos a$;
- $\sin(a - b) = \sin a \cos b - \sin b \cos a$;
- $\cos(a + b) = \cos a \cos b - \sin a \sin b$;
- $\cos(a - b) = \cos a \cos b + \sin a \sin b$.

The tangent and cotangent function

Function	Notation	Domain	Parity	Period	Range
Tangent function	$\tan x = \frac{\sin x}{\cos x}$	$\mathbb{R} - \{A = \frac{\pi}{2} + 2k\pi\}$	Odd	$T = \pi$	$\mathbb{R}$
Cotangent function	$\cot x = \frac{\cos x}{\sin x}$	$\mathbb{R} - \{B = k\pi\}$	Odd	$T = \pi$	$\mathbb{R}$

We have the equality $\cot x = \frac{1}{\tan x}$; $\forall x \in \mathbb{R} - \{A \cup B\}$.

Addition formula

- $\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$
- $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

Derivatives and variations

$(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x > 0, \forall x \in \mathbb{R} - \{A\}$, so, $\tan$ is strictly increasing.

$(\cot x)' = \frac{-1}{\sin^2 x} = -(1 + \cot^2 x) < 0, \forall x \in \mathbb{R} - \{B\}$, so, $\tan$ is strictly decreasing.

Graphs**2.2.2 Solution of trigonometric Equations**

$$1. \sin x = \sin a \Rightarrow \begin{cases} x = a + 2k\pi \\ x = \pi - a + 2k\pi \end{cases} \quad k \in \mathbb{Z}$$

$$2. \cos x = \cos a \Rightarrow \begin{cases} x = a + 2k\pi \\ x = -a + 2k\pi \end{cases} \quad k \in \mathbb{Z}$$

3. $\tan x = \tan a \Rightarrow x = a + k\pi; \quad k \in \mathbb{Z}.$

4. if $\sin^2 x = \sin^2 a$ or $\cos^2 x = \cos^2 a$ or $\tan^2 x = \tan^2 a \Rightarrow \begin{cases} x = a + k\pi \\ x = -a + k\pi \end{cases} \quad k \in \mathbb{Z}.$

2.3 Hyperbolic Function

2.3.1 Hyperbolic sine and cosine function

Definition 2.3.1 Given a real x , we call hyperbolic sine of x , denoted $\sinh(x)$, the function defined by

$$\begin{aligned} \sinh &: \mathbb{R} \mapsto \mathbb{R} \\ x &\mapsto \sinh(x) = \frac{e^x - e^{-x}}{2}. \end{aligned}$$

Definition 2.3.2 Given a real x , we call hyperbolic cosine of x , denoted $\cosh(x)$, the function defined by

$$\begin{aligned} \cosh &: \mathbb{R} \mapsto \mathbb{R} \\ x &\mapsto \cosh(x) = \frac{e^x + e^{-x}}{2}. \end{aligned}$$

Proposition 2.3.1 1. We have

$$\cosh^2 x - \sinh^2 x = \frac{e^{2x} + e^{-2x} + 2}{4} - \frac{e^{2x} + e^{-2x} - 2}{4} = 1,$$

thus

$$\cosh^2 - \sinh^2 = 1$$

2. $\sinh$ and $\cosh$ are continuous and differentiable on $\mathbb{R}$, where

$$\forall x \in \mathbb{R}, (\sinh x)' = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\forall x \in \mathbb{R}, (\cosh x)' = \frac{e^x - e^{-x}}{2} = \sinh x$$

3. For all $x \in \mathbb{R}$, we have $\sinh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -\sinh(x)$, so $\sinh$ is odd, and

$$\cosh(-x) = \frac{e^x - e^{-x}}{2} = \frac{e^{-x} + e^x}{2} = \cosh(x), \text{ so } \cosh \text{ is even.}$$

1. We have $(\sinh x)' > 0$, then $\sinh$ is strictly increasing on $\mathbb{R}$,

$$\lim_{x \rightarrow -\infty} \sinh(x) = -\infty; \quad \lim_{x \rightarrow +\infty} \sinh x = +\infty, \quad \sinh(0) = 0$$

$$\begin{array}{ccccccc} x & & -\infty & & 0 & & +\infty \\ (\sinh x)' & & & & + & & \\ & & & & & & +\infty \\ \sinh(x) & & & & \nearrow & & \\ & & & & -\infty & & \end{array}$$

so

$$\text{Rang}(\sinh(x)) = \mathbb{R}.$$

and $(\cosh(x))' = \sinh(x) = 0 \Leftrightarrow x = 0$

a) If $x > 0 \Leftrightarrow (\cosh(x))' > 0$,

so, $\cosh$ is strictly increasing on $[0; +\infty[$. b) If $x < 0 \Leftrightarrow (\cosh(x))' < 0$,

so, $\cosh$ is strictly increasing on $] -\infty; 0]$.

$$\lim_{x \rightarrow -\infty} \cosh(x) = +\infty; \quad \lim_{x \rightarrow +\infty} \cosh x = +\infty, \quad \cosh(0) = 1$$

so

$$\text{Rang}(\cosh(x)) = [1; +\infty[.$$

2.3.2 Hyperbolic tangent and cotangent function

Definition 2.3.3 Given a real x , we call the hyperbolic tangent of x denoted $\tanh(x)$, the function defined by

$$\begin{aligned} \tanh &: \mathbb{R} \mapsto]-1; 1[\\ x &\mapsto \tanh(x) = \frac{\sinh(x)}{\cosh(x)}. \end{aligned}$$

Definition 2.3.4 Given a real x , we call the hyperbolic cotangent of x denoted $\coth(x)$, the function defined by

$$\begin{aligned} \coth &: \mathbb{R}^* \mapsto]-\infty; -1[\cup]1; +\infty[\\ x &\mapsto \coth(x) = \frac{\cosh(x)}{\sinh(x)}. \end{aligned}$$

Proposition 2.3.2 1. We have $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$.

2. For $x \neq 0$, We have $\coth(x) = \frac{1}{\tanh(x)} = \frac{e^{2x} + 1}{e^{2x} - 1}$.

3. $\tanh$ is continuous and differentiable on $\mathbb{R}$:

$$(\tanh(x))' = \left(\frac{\sinh(x)}{\cosh(x)} \right)' = \frac{1}{\cosh^2(x)} = 1 - \tanh^2(x).$$

4. $\coth$ is continuous and differentiable on $\mathbb{R}^*$:

$$(\coth(x))' = \left(\frac{\cosh(x)}{\sinh(x)} \right)' = \frac{-1}{\sinh^2(x)} = 1 - \coth^2(x).$$

5. We have $\forall x \in \mathbb{R}, \tanh(-x) = -\tanh(x)$ and $\forall x \in \mathbb{R}^*, \coth(-x) = -\coth(x)$; then, $\tanh$ and $\cosh$ are odd functions.

6. $\forall x \in \mathbb{R}$, We have $(\tanh(x))' > 0$,
so, $\tanh$ is strictly increasing on $\mathbb{R}$.

$$\lim_{x \rightarrow -\infty} \tanh(x) = -1; \quad \lim_{x \rightarrow +\infty} \tanh(x) = 1, \quad \tanh(0) = 0$$

so

$$\text{Rang}(\tanh(x)) =]-1; 1[.$$

7. $\forall x \in \mathbb{R}^*$ We have $(\coth(x))' < 0$,
so, $\coth$ is strictly decreasing on $\mathbb{R}^*$.

$$\lim_{x \rightarrow -\infty} \coth(x) = -1; \quad \lim_{x \rightarrow +\infty} \coth(x) = 1,$$

$$\lim_{x \rightarrow 0^+} \coth(x) = +\infty; \quad \lim_{x \rightarrow 0^-} \coth(x) = -\infty,$$

so

$$\text{Rang}(\coth(x)) =]-\infty; -1[\cup]1; +\infty[.$$