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0.1 Continuous functions

0.1.1 Continuouty at a point

Definition 0.1.1 *Let $I \subseteq \mathbb{R}$ and $f : I \mapsto \mathbb{R}$, be a function and let $x_0 \in I$, we say that f is continuous at x_0 if*

$$\lim_{x \to x_0} f(x) = f(x_0).$$

0ne-sided continuity

- f is left continuous at a point x_0, if $\lim_{x \to x_0^-} f(x) = f(x_0)$.
- f is right continuous at a point x_0, if $\lim_{x \to x_0^+} f(x) = f(x_0)$.
- f is continuous at a point x_0, $\Leftrightarrow$ $\lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x) = f(x_0)$.

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0.1.2 Continuity on a interval

Definition 0.1.2 *A function f is continuous over an interval I, if it is continuous at every point in the interval.*

More precisely:

- f is continuous on the open interval $]a;b[$, if f is continuous at every point in the interval $]a;b[$.
- f is continuous on the closed interval $[a;b]$, if f is continuous on $]a;b[$, continuous form the right at a, and continuous form the left at b.

Remark 0.1.1 .

- *Every polynomial function is continuous on* $\mathbb{R}$.
- *Every rational function is continuous on its domain.*
- sin *and* cos *are continuous every where on* $\mathbb{R}$.
- *The square root is continuous on* $\mathbb{R}^+$.

0.1.3 Intermediate Value Theorem (IVT)

Theorem 0.1.1 (Intermediate Value Theorem) *Let $f : [a,b] \to \mathbb{R}$ be a function such that*

- *f is continuous on the closed interval $[a,b]$.*
- *Let k be any number between $f(a)$ and $f(b)$.*

Then, there exists at least one $c \in (a,b)$ such that

$$f(c) = k.$$

Theorem 0.1.2 (Intermediate Value Theorem – Common Form) *Let $f : [a,b] \to \mathbb{R}$ be a function such that*

- *f is continuous on the closed interval $[a,b]$.*
- $f(a)f(b) < 0$.

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Then, there exists at least one $c \in (a,b)$ such that

$$f(c) = 0.$$

Example 1 (1) We will show that the equation $x^5 - 3x + 1 = 0$ has at least one root in $(0,1)$. For this, we define the continuous function $f : [0,1] \to \mathbb{R}$ as follows:

$$f(x) = x^5 - 3x + 1.$$

It follows that

$$f(0) = 1 > 0 \quad \text{and} \quad f(1) = -1 < 0.$$

So, as a result of the Intermediate Value Theorem, there exists at least one $c \in (0,1)$ such that

$$f(c) = c^5 - 3c + 1 = 0.$$

As a consequence, the equation $x^5 - 3x + 1 = 0$ has at least one root in $(0,1)$.

Example 2 Show that the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has a solution in the interval $[1,2]$.

Consider the function

$$f(x) = 4x^3 - 6x^2 + 3x - 2$$

over the closed interval $[1,2]$.

The function f is a polynomial, therefore it is continuous on $[1,2]$.

We have

$$f(1) = -1 \quad \text{and} \quad f(2) = 12,$$

hence $f(1)\,f(2) < 0$.

By the Intermediate Value Theorem, there exists a value $c \in (1,2)$ such that

$$f(c) = 0,$$

that is, there exists at least one solution to the equation $f(x) = 0$ in the interval $(1,2)$.

Example 3 Show that the equation $x^7 - x^2 + 1 = 0$, has at least one solution in the interval $I = [-2, 0]$.

Let $f(x) = x^7 - x^2 + 1$. f is continuous on $\mathbb{R}$ (polynomial), then f is continuous on $I = [-2, 0]$.

$f(-2) = -123 < 0$ and $f(0) = 1 > 0 \Leftrightarrow f(-2)\,f(1) < 0$.

Therefore by IVT the equation $f(x) = 0$ has at least one solution $\alpha \in]-2;0[$, such that $f(\alpha) = 0$.

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0.1.4 Inverse functions and the monotonic bijection theorem

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Theorem 0.1.3 *Let $f : I \to \mathbb{R}$ be a function defined on a closed interval I and suppose that f is monotonic on I. Assume that f is continuous and strictly monotonic on I. Then:*

1. *$f : I \to f(I)$ is bijective.*
2. *The inverse function $f^{-1} : f(I) \to I$ exists, and it is continuous and strictly monotonic on $f(I)$.*

Example 0.1.1 *Let the function*

$$f :]-\infty, 0] \to \mathbb{R}, \quad x \mapsto f(x) = x^2$$

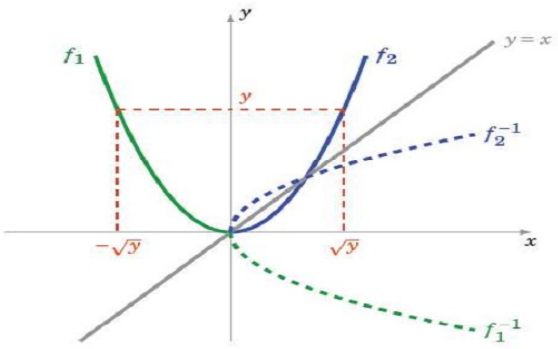
The function f is continuous and strictly decreasing on $]-\infty, 0]$, according to the bijection theorem.

$$f :]-\infty, 0] \to f(]-\infty, 0]) = [0, +\infty[$$

is bijective, and

$$f^{-1} : [0, +\infty[\to]-\infty, 0], \quad y \mapsto x = -\sqrt{y}$$

is continuous and strictly monotonic (decreasing).



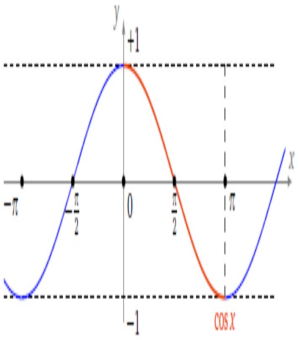
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0.2 Applications: inverse trigonometric and hyperbolic functions

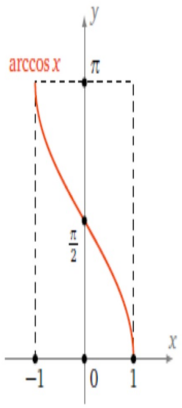
0.2.1 The Arccosine Function

The cosine function $\cos x$ is continuous and strictly decreasing on the interval $[0,\pi]$, so, $cos : [0;\pi] \longrightarrow [-1;1]$ is a bijection Therefore, it admits an inverse function defined by

$$\arccos : [-1,1] \longrightarrow [0,\pi].$$



Graph of the function cos



Graph of the function arccos

Properties

1. arccos is continuous and strictly decreasing on $[-1;1]$.
2. arccos is differentiable on $]-1;1[$ and

$$\forall x \in]-1;1[\quad (\arccos(x))' = -\frac{1}{\sqrt{1-x^2}}.$$

More generally:

$$\forall x \in]-1;1[\quad (\arccos(f(x)))' = -\frac{f'(x)}{\sqrt{1-(f(x))^2}}.$$

3. $\forall x \in [-1,1] \quad \cos(\arccos(x)) = x,$
4. $\forall x \in [0,\pi] \quad \arccos(\cos x) = x,$

5. $\forall x \in]-1;1[\quad \sin(\arccos x) = \sqrt{1-x^2}$.

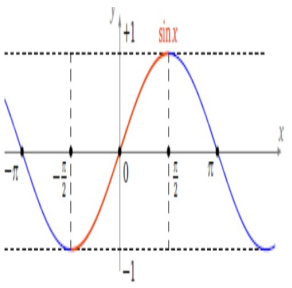
Example 0.2.1 $\sin(\arccos(\frac{3}{5})) = \sqrt{1-(\frac{3}{5})^2} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

$\sin(\arccos(\frac{2}{5}))$

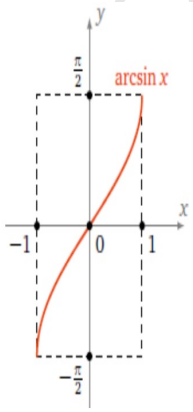
0.2.2 The Arcsine Function

The sine function $\sin x$ is continuous and strictly increasing on the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$, so, $sin : [\frac{-\pi}{2}, \frac{\pi}{2}] \longrightarrow [-1;1]$ is a bijection Therefore, it admits an inverse function defined by

$$\arcsin : [-1;1] \longrightarrow [\frac{-\pi}{2}, \frac{\pi}{2}].$$



Graph of function sin



Graph of function arcsin

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Properties

1. arcsin is continuous and strictly increasing on $[-1;1]$.
2. arccos is differentiable on $]-1;1[$ and

$$\forall x \in]-1;1[\quad (\arcsin(x))' = \frac{1}{\sqrt{1-x^2}}.$$

More generally:

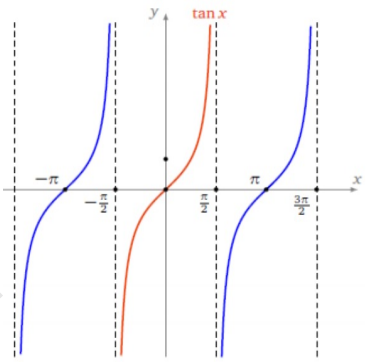
$$\forall x \in]-1;1[\quad (\arcsin(f(x)))' = \frac{f'(x)}{\sqrt{1-(f(x))^2}}.$$

3. $\forall x \in [-1,1] \quad \sin(\arcsin(x)) = x,$
4. $\forall x \in [\frac{-\pi}{2}, \frac{\pi}{2}] \quad \arcsin(\sin x) = x,$
5. $\forall x \in]-1;1[\quad \cos(\arcsin x) = \sqrt{1-x^2}.$

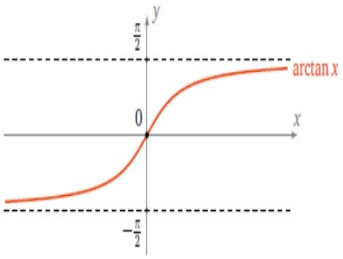
0.2.3 The Arctan Function

The function tan is continuous and strictly increasing on the interval $]\frac{-\pi}{2}, \frac{\pi}{2}[$, so, $tan :]\frac{-\pi}{2}, \frac{\pi}{2}[\longrightarrow \mathbb{R}$ is a bijection Therefore, it admits an inverse function defined by

$$\arctan : \mathbb{R} \longrightarrow]\frac{-\pi}{2}, \frac{\pi}{2}[.$$



Graph of function tan



Graph of function arctan

Properties

1. arctan is continuous and strictly increasing on $\mathbb{R}$.

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2. arctan is differentiable on $\mathbb{R}$ and

$$\forall x \in \quad (\arctan(x))' = \frac{1}{1+x^2}.$$

More generally:

$$\forall x \in \mathbb{R} \quad (\arctan(f(x)))' = \frac{f'(x)}{1+[f(x)]^2}.$$

3. $\forall x \in \mathbb{R} \quad \tan(\arctan(x)) = x,$

4. $\forall x \in]\frac{-\pi}{2}, \frac{\pi}{2}[\quad \arctan(\tan x) = x,$

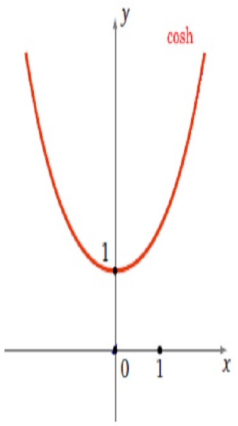
5. $\forall x \in \mathbb{R} \quad \cos^2(\arctan x) = \frac{1}{1+x^2}.$

Remark 0.2.1 $arctan(x) \neq \frac{arcsin(x)}{arccos(x)}$

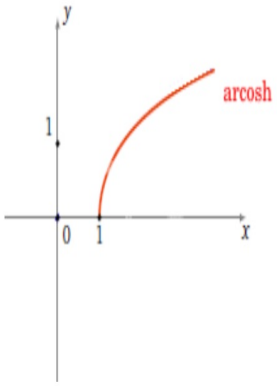
0.2.4 Inverse hyperbolic cosine Function

The function cosh is continuous and strictly increasing on the interval $[0,+\infty[$, so, $cosh : [0,+\infty[\longrightarrow [1,+\infty[$ is a bijection Therefore, it admits an inverse function defined by

$$\text{arcosh} : [1,+\infty[\longrightarrow [0,+\infty[.$$



Graph of function cosh



Graph of function arcosh

Properties

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1. arcosh is continuous and strictly increasing on $[1;+\infty[$.

2. arcosh is differentiable on $]1;+\infty[$ and

$$\forall x \in]1;+\infty[\quad (\operatorname{arcosh}(x))' = \frac{1}{\sqrt{x^2-1}}.$$

3. The function arcosh is neither even nor odd because its domain is $[1;+\infty[$.

4. $\forall x \in [1,+\infty[\quad \cosh(\operatorname{arcosh}(x)) = x$,

5. $\forall x \in [0,+\infty] \quad \operatorname{arcosh}(\cosh(x)) = x$,

6. $\forall x \in]1;+\infty[\quad \sinh(\operatorname{arcosh}(x)) = \sqrt{x^2-1}$.

Lagarithmic expression $\forall x \in [1,+\infty[\quad \operatorname{arcosh}(x) = \ln(x+\sqrt{x^2-1})$.

Example 0.2.2 *Solve the equation using two different methods*

$$\operatorname{arcosh}(x) = \frac{5}{2}.$$

Method 1

We have $\operatorname{arcosh}(x) = \ln(x+\sqrt{x^2-1}) = \frac{5}{2} \Leftrightarrow x+\sqrt{x^2-1} = e^{\frac{5}{2}} \Leftrightarrow \sqrt{x^2-1} = e^{\frac{5}{2}} - x$ *such that* $x \in [1;e^{\frac{5}{2}}]$

$\Leftrightarrow x^2-1 = x^2+e^5-2xe^{\frac{5}{2}} \Leftrightarrow x = \dfrac{e^5+1}{2e^{\frac{5}{2}}} = \dfrac{e^{\frac{5}{2}}+e^{\frac{-5}{2}}}{2} \in [1;e^{\frac{5}{2}}]$

Method 2: $\operatorname{arcosh}(x) = \frac{5}{2} \Leftrightarrow x = \cosh(\frac{5}{2}) \Leftrightarrow x = \dfrac{e^{\frac{5}{2}}+e^{\frac{-5}{2}}}{2}$

0.2.5 Inverse hyperbolic sine Function

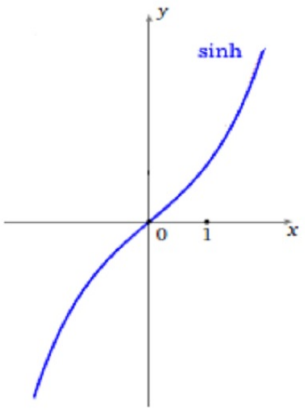
The function sinh is continuous and strictly increasing on $\mathbb{R}$, so, $sinh : \mathbb{R} \longrightarrow \mathbb{R}$ is a bijection Therefore, it admits an inverse function defined by

$$\operatorname{arsinh} : \mathbb{R} \longrightarrow \mathbb{R}.$$

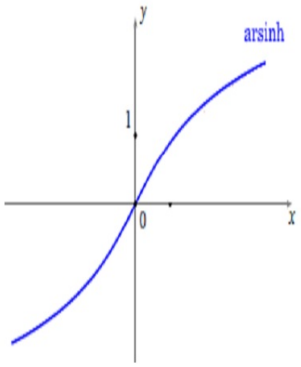
Properties

1. arsinh is continuous and strictly increasing on $\mathbb{R}$.

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Graph of function sinh



Graph of function arsinh

2. arsinh is differentiable on $\mathbb{R}$ and

$$\forall x \in \mathbb{R} \quad (\text{arsinh}(x))' = \frac{1}{\sqrt{x^2+1}}.$$

3. The function arsinh is odd.

4. $\forall x \in \mathbb{R} \quad \sinh(\text{arsinh}(x)) = x,$

5. $\forall x \in \mathbb{R} \quad \text{arsinh}(\sinh(x)) = x,$

6. $\forall x \in \mathbb{R} \quad \cosh(\text{arsinh}(x)) = \sqrt{x^2+1}.$

Lagarithmic expression $\forall x \in [1,+\infty[\quad \text{arsinh}(x) = \ln(x+\sqrt{x^2+1}).$

Example 0.2.3 *Solve the equation using two different methods*

$$\text{arsinh}(x) = \frac{5}{2}.$$

Method 1

We have $\text{arsinh}(x) = \ln(x+\sqrt{x^2+1}) = \frac{5}{2} \Leftrightarrow x+\sqrt{x^2-1} = e^{\frac{5}{2}} \Leftrightarrow \sqrt{x^2-1} = e^{\frac{5}{2}} - x$ *such that* $x \in [1; e^{\frac{5}{2}}]$

$\Leftrightarrow x^2-1 = x^2+e^5-2xe^{\frac{5}{2}} \Leftrightarrow x = \dfrac{e^{\frac{5}{2}}-e^{\frac{-5}{2}}}{2} \in]-\infty; e^{\frac{5}{2}}]$

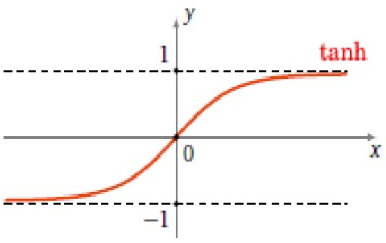
Method 2: $\text{arsinh}(x) = \frac{5}{2} \Leftrightarrow x = \sinh(\frac{5}{2}) \Leftrightarrow x = \dfrac{e^{\frac{5}{2}}-e^{\frac{-5}{2}}}{2}$

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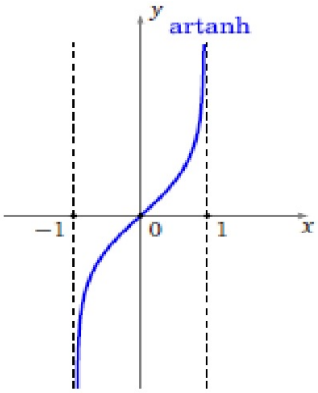
0.2.6 Inverse hyperbolic tangent Function

The function tanh is continuous and strictly increasing on $\mathbb{R}$, so, $tanh : \mathbb{R} \longrightarrow]-1;1[$ is a bijection Therefore, it admits an inverse function defined by

$$\text{artanh} :]-1;1[\longrightarrow \mathbb{R}.$$



Graph of function tanh



Graph of function artanh

Properties

1. artanh is continuous and strictly increasing on $]-1;1[$.
2. artanh is differentiable on $]-1;1[$ and

$$\forall x \in]-1;1[\quad (\text{artanh}(x))' = \frac{1}{1-x^2}.$$

3. The function artanh is odd.
4. $\forall x \in]-1;1[\quad \tanh(\text{artanh}(x)) = x,$
5. $\forall x \in \mathbb{R} \quad \text{artanh}(\tanh(x)) = x,$
6. $\forall x \in]-1;1[\quad \cosh(\text{arsinh}(x)) = \frac{1}{\sqrt{1-x^2}}.$

Lagarithmic expression $\forall x \in]-1,1[\quad \text{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).$

Example 0.2.4 *Solve* $\in]-1,1[$ *the equation*

$$\operatorname{artanh}(x) = \ln 3.$$

We have $\operatorname{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln(3) \Leftrightarrow \ln\left(\frac{1+x}{1-x}\right) = \ln(9) \Leftrightarrow \frac{1+x}{1-x} = 9$
$\Leftrightarrow x = \frac{8}{10} = \frac{4}{5}$

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